First Name: ____________________ ID#____________________

Last Name: ____________________

Section: ____________________ $= \begin{cases} 1a & \text{Tuesday with S. Kim} \\ 1b & \text{Thursday with S. Kim} \\ 1c & \text{Tuesday with J. Murphy} \\ 1d & \text{Thursday with J. Murphy} \\ 1e & \text{Tuesday with F. Robinson} \\ 1f & \text{Thursday with F. Robinson} \end{cases}$

Rules:

- There are **NINE** problems for a total of 100 points.
- Use the backs of the pages.
- No calculators, computers, notes, books, e.t.c..
- Out of consideration for your classmates, no chewing, humming, pen-twirling, snoring, e.t.c.. Try to sit still.
- Turn off your cell-phone.

1	2	3	4	$\sum$

(1) **(10 points)** Answer by filling in the blanks or indicating TRUE / FALSE, as appropriate.

(1) The initial-value problem

$$\frac{d}{dt}x = x^{1/2} \quad \text{with} \quad x(0) = 0$$

has no solution since $\frac{d}{dx}(x^{1/2})$ is discontinuous at $x = 0$. **TRUE / FALSE**

(2) Consider the following first order differential equation

$$\frac{d}{dt}x = \sqrt{x^2 - 9}.$$

The existence and uniqueness theorem (of Picard) guarantees that this equation admits a unique solution passing through the point

(a) $(1, 4)$ **TRUE / FALSE**

(b) $(2, -3)$ **TRUE / FALSE**

(c) $(5, 3)$ **TRUE / FALSE**

(d) $(-1, 1)$ **TRUE / FALSE**

(3) A differential operator that annihilates $e^{2t}(t + t^2)$ is ________________________

(4) The only solution to the initial-value problem

$$x'' + t^2 x = 0 \quad \text{with} \quad x(0) = 0 \quad \text{and} \quad x'(0) = 0$$

is ________________________

(5) A fundamental set of solutions to the equation

$$(t - 2)x'' + x = 0$$

exists on any time interval not containing the point ________________________

(6) A vector v in $\mathbb{R}^n$ with all entries equal to zero is *never* an eigenvector of an $n \times n$ matrix A. **TRUE / FALSE**.

(7) The following equation is exact:

(a) $(2x - 1)\,dx + (3y + 7)\,dy = 0.$ **TRUE / FALSE**

(b) $\left(1 + \ln(x) + \frac{y}{x}\right)dx = (1 - \ln(x))\,dy.$ **TRUE / FALSE**

(c) $(y^3 - y^2\sin(x) - x)\,dx + (3xy^2 + 2y\sin(x))\,dy = 0.$ **TRUE / FALSE**

(2) **(10 points)**

(a) Find an integrating factor depending only on x to make the following equation exact on the rectangle $(-\infty, \infty) \times (-\infty, \infty)$:

$$\left(2 - \frac{y \sin(xy)}{x} - 5\frac{y^4}{x}\right) dx - \left(20y^3 + \sin(xy)\right) dy = 0.$$

(b) Find the solution to this equation.

(3) **(10 points)** Consider the differential equation

$$\frac{dx}{dt} = (x+1)x(1-x).$$

(a) What is the largest rectangle in the tx plane on which you can apply the existence and uniqueness (Picard) theorem? Justify your answer.

(b) Identify the equilibrium points.

(c) Draw a phase diagram and identify the stable and unstable points.

(d) Sketch the equilibrium solutions in the tx plane. These equilibrium solutions divide the tx plane into regions. Sketch at least one solution curve in each of these regions.

(e) For the particular solution with initial condition $x(0) = -0.4$, what is the limit $\lim_{t\to\infty} x(t)$?

(4) **(10 points)**
Find the general solution to the equation

$$x'' + 2x' + x = e^{-t} \ln(t) \quad \text{for} \quad t > 0.$$

(5) **(10 points)** Consider the equation

$$(2 - t^2)x'' + 2tx' - 2x = 0.$$

(a) Verify that $\phi_1(t) = t$ is a solution to the equation.
(b) Let ϕ_2 be a second solution to the differential equation so that $W(\phi_1, \phi_2)(0) = 1$. Use Abel's theorem to find the Wronskian determinant of ϕ_1 and ϕ_2 at all times $t \in (-\sqrt{2}, \sqrt{2})$.
(c) Use part (b) to find all possible solutions ϕ_2 satisfying $W(\phi_1, \phi_2)(0) = 1$.

(6) **(10 points)**

(a) Find the general solution to

$$\mathbf{x}' = A\mathbf{x} \qquad \text{with} \qquad A = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}.$$

(b) Find the exponential matrix e^{tA}.

(7) **(10 points)** Let

$$A = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix}.$$

(a) Find the eigenvalues of A. What are their algebraic and geometric multiplicities?

(b) Find the general solution to the system $\mathbf{x}' = A\mathbf{x}$.

(8) **(16 points)** For each of the following matrices

$$(1)\quad A = \begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix} \qquad\qquad (2)\quad A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

perform the following tasks:

(a) Identify all equilibrium points of the system $\mathbf{x}' = A\mathbf{x}$ and plot them in the phase plane.

(b) Write down the general solution to the system $\mathbf{x}' = A\mathbf{x}$.

(c) Draw the phase plane portraits making sure to identify the direction of motion.

(d) Name the type of equilibrium point at the origin; specify whether it is stable, unstable, or asymptotically stable.

(9) **(14 points)**

(a) Let A be an $n \times n$ matrix with constant real entries. Define what it means for a number λ to be an eigenvalue of A.

(b) Define what it means for two vectors $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$ to be linearly independent.

(c) Let A be an $n \times n$ matrix with constant real entries and let λ_1 and λ_2 be two *distinct* eigenvalues of A. Assume $v_1 \in \mathbb{R}^n$ is an eigenvector with eigenvalue λ_1 and $v_2 \in \mathbb{R}^n$ is an eigenvector with eigenvalue λ_2. Show that v_1 and v_2 are linearly independent.