

## HOMWORK 2

- Section 2.2 in the book: Exercises 6, 12, 16, 34, 42.
- Section 2.4 in the book: Exercises 4, 16, 22, 28, 40.
- Section 2.5 in the book: Exercises 8, 12.

**Problem 1.** (a) Let  $a \in \mathbb{R}$  be a constant and let  $b_1 : \mathbb{R} \rightarrow \mathbb{R}$  and  $b_2 : \mathbb{R} \rightarrow \mathbb{R}$  be two continuous functions such that  $b_1(t) \leq b_2(t)$  for all  $t \geq 0$ . Consider the linear differential equations

$$\frac{dx}{dt} = ax + b_1(t) \quad \text{and} \quad \frac{dy}{dt} = ay + b_2(t).$$

Assuming that  $x(0) = y(0)$ , show that  $x(t) \leq y(t)$  for all  $t \geq 0$ .

(b) Show that if  $z$  is a differentiable function that satisfies  $\frac{dz}{dt} \leq az$ , then

$$z(t) \leq z(0)e^{at} \quad \text{for all } t \geq 0.$$

**Problem 2.** Beer containing 6% alcohol per gallon is pumped into a vat that initially contains 400 gallons of beer at 3% alcohol. The rate at which the beer is pumped in is 3 gallons per minute, whereas the mixed liquid is pumped out at a rate of 4 gallons per minute. Find the number of gallons of alcohol  $A(t)$  in the tank at any time. What is the percentage of alcohol in the tank after 60 minutes? When is the tank empty?

**Problem 3.** When forgetfulness is taken into account, the rate of memorization of a subject is given by

$$\frac{dA}{dt} = k_1(M - A) - k_2A,$$

where  $k_1 > 0$ ,  $k_2 > 0$ ,  $A(t)$  is the amount memorized at time  $t$ ,  $M$  is the total amount to be memorized, and  $M - A$  is the amount remaining to be memorized. Solve for  $A(t)$  and graph the solution, assuming that  $A(0) = 0$ . Find the limiting value of  $A$  as  $t \rightarrow \infty$ .