HOMEWORK 1

Due on Monday, March 1st, in class.

Exercise 1. Let $d = 3$ and $1 < p < 2$. Prove that the initial value problem

$$u_{tt} - \Delta u \pm |u|^p u = 0$$
$$u(0) = u_0, \quad u_t(0) = u_1$$

is locally wellposed for initial data $(u_0, u_1) \in H^1_x \times L^2_x$.

Exercise 2. (Standard blowup criterion) Let $d = 3$ and $2 \leq p \leq 4$. Let initial data (u_0, u_1) belong to the critical homogeneous space $\dot{H}^{s_c}_x \times \dot{H}^{s_c - 1}_x$. (Recall $s_c = \frac{3}{2} - \frac{2}{p}$.) Let $u : [0, T_0) \times \mathbb{R}^3 \to \mathbb{R}$ be the unique strong solution to

$$u_{tt} - \Delta u \pm |u|^p u = 0$$
$$u(0) = u_0,\ u_t(0) = u_1.$$

Assume that

$$\|u\|_{L^{2p}_{t,x}([0,T_0)\times\mathbb{R}^3)} < \infty.$$

Prove that there exists $\delta > 0$ such that u extends to a strong solution on $[0, T_0 + \delta]$.

Exercise 3. (Scattering, part 1) Let $d = 3$ and $2 \leq p \leq 4$. Let $(u_0, u_1) \in \dot{H}^{s_c}_x \times \dot{H}^{s_c-1}_x$. Prove that there exists a unique solution $u : [T, \infty) \times \mathbb{R}^3 \to \mathbb{R}$ to

$$u_{tt} - \Delta u \pm |u|^p u = 0$$

for some $T \in \mathbb{R}$ such that $(u, u_t) \in C_t([T, \infty); \dot{H}^{s_c}_x \times \dot{H}^{s_c-1}_x)$, $u \in L^{2p}_{t,x}([T, \infty) \times \mathbb{R}^3)$ and

$$\left\|u(t) - \cos(t|\nabla|)u_0 - \tfrac{\sin(t|\nabla|)}{|\nabla|}u_1\right\|_{\dot{H}^{s_c}_x} \to 0 \quad \text{as} \quad t \to \infty.$$

Exercise 4. (Scattering, part 2) Let $d = 3$ and $2 \leq p \leq 4$. Let $(u_0, u_1) \in \dot{H}^{s_c}_x \times \dot{H}^{s_c-1}_x$. Prove that if

$$\|(u_0, u_1)\|_{\dot{H}^{s_c}_x \times \dot{H}^{s_c-1}_x} \leq \eta$$

for $\eta = \eta(p)$ sufficiently small, then there exists a unique global solution u to

$$u_{tt} - \Delta u \pm |u|^p u = 0$$
$$u(0) = u_0,\ u_t(0) = u_1$$

satisfying $(u, u_t) \in C_t(\dot{H}^{s_c}_x \times \dot{H}^{s_c-1}_x)$ and $u \in L^{2p}_{t,x}(\mathbb{R} \times \mathbb{R}^3)$. Moreover, for such small initial data there exist $(u_0^\pm, u_1^\pm) \in \dot{H}^{s_c}_x \times \dot{H}^{s_c-1}_x$ so that

$$\left\|u(t) - \cos(t|\nabla|)u_0^\pm - \tfrac{\sin(t|\nabla|)}{|\nabla|}u_1^\pm\right\|_{\dot{H}^{s_c}_x} \to 0 \quad \text{as} \quad t \to \pm\infty.$$

Exercise 5. (Domain of dependence for classical solutions) Let I be a time interval containing 0 and let $u, v \in C^2_{t,x}(I \times \mathbb{R}^3)$ be two solutions to

$$w_{tt} - \Delta w \pm |w|^p w = 0, \quad p > 0$$

with initial data (u_0, u_1) and (v_0, v_1), respectively. Prove that if the initial data agree on a ball $\{x \in \mathbb{R}^3 : |x - x_0| < R\}$, then the solutions agree on $\{(t, x) \in I \times \mathbb{R}^3 : |x - x_0| < R - |t|\}$.