HOMEWORK

Problem 1. Let $d \geq 1$. Using Gaussian wave packets prove that for any $\varepsilon > 0$,

$$\sup_{f \in \mathcal{S}(\mathbb{R}^d), f \neq 0} \frac{\int_{\mathbb{R}} \int_{|x| \leq 1} |(e^{it\Delta} f)(x)|^2 \, dx \, dt}{\|\langle \nabla \rangle^{-1/2-\varepsilon} f\|_2^2} = \infty.$$

Problem 2. (Subcritical unconditional uniqueness) Let $d \geq 1$, $p > 0$, and $0 \leq s \leq 1$ such that $s > s_c := \frac{d}{2} - \frac{2}{p}$. Let $u : [0,T] \times \mathbb{R}^d \to \mathbb{C}$ be the strong solution (constructed in class) to

$$iu_t + \Delta u = \pm |u|^p u \quad \text{with} \quad u(0) = u_0 \in H^s(\mathbb{R}^d).$$

Let $v \in C_t H_x^s([0,T] \times \mathbb{R}^d)$ be such that

$$v(t) = e^{it\Delta} u_0 \mp i \int_0^t e^{i(t-s)\Delta} (|u|^p u)(s) \, ds \quad \text{for all} \quad t \in [0,T].$$

Prove that $u(t) = v(t)$ a.e. for all $t \in [0,T]$ in the following cases:

- $1 = d > 2s$ and $p \leq \frac{1+2s}{1-2s}$
- $d \geq 2$, $d > 2s$, and $p \leq \frac{2+2s}{d-2s}$, with $p < \frac{2+2s}{d-2s}$ if $d = 2$ or $s = 1$.

Problem 3. (Critical blowup criterion) Let $d \geq 1$ and $p > 0$ such that the critical regularity satisfies $0 \leq s_c \leq 1$. Let initial data $u_0 \in H^{s_c}(\mathbb{R}^d)$ and let $u : [0, T_0) \times \mathbb{R}^d \to \mathbb{C}$ be the unique strong solution to

$$iu_t - \Delta u = \pm |u|^p u \quad \text{with} \quad u(0) = u_0.$$

Assume that

$$\|u\|_{L_{t,x}^{\frac{p(d+2)}{2}}([0,T_0) \times \mathbb{R}^d)} < \infty.$$

Prove that there exists $\delta > 0$ such that u extends to a strong solution on $[0, T_0 + \delta]$.

Problem 4. (Existence of wave operators) Let $d \geq 1$ and $p > 0$ such that the critical regularity satisfies $0 \leq s_c \leq 1$. Let $u_+ \in H^{s_c}(\mathbb{R}^d)$. Prove that there exist $T \in \mathbb{R}$ and a unique strong solution $u : [T, \infty) \times \mathbb{R}^d \to \mathbb{C}$ to

$$iu_t - \Delta u = \pm |u|^p u$$

such that

$$\left\| u(t) - e^{it\Delta} u_+ \right\|_{H^{s_c}(\mathbb{R}^d)} \to 0 \quad \text{as} \quad t \to \infty.$$

Problem 5. For any integer $k \geq 0$, let the weighted Sobolev space $H^{k,k}(\mathbb{R}^d)$ be the closure of Schwartz functions under the norm

$$\|f\|_{H^{k,k}(\mathbb{R}^d)} := \sum_{|\alpha| + |\beta| \leq k} \left\| x^\alpha \partial_x^\beta f \right\|_{L^2(\mathbb{R}^d)},$$

where α, β are multi-indices of length d.

1) Prove that

$$\|e^{it\Delta} f\|_{H^{k,k}(\mathbb{R}^d)} \lesssim_{k,d} \langle t \rangle^k \|f\|_{H^{k,k}(\mathbb{R}^d)}.$$

2) Show that if $k > \frac{d}{2}$, then
$$\|fg\|_{H^{k,k}(\mathbb{R}^d)} \lesssim_{k,d} \|f\|_{H^{k,k}(\mathbb{R}^d)} \|g\|_{H^{k,k}(\mathbb{R}^d)}.$$

Problem 6. (Classical NLS solutions) Let $d \geq 1$, $k > \frac{d}{2}$, and let $p > 0$ be an even integer. Then for any $R > 0$ there exists $T = T(d,k,p,R) > 0$ such that for all $u_0 \in \{f \in H^{k,k}(\mathbb{R}^d) : \|f\|_{H^{k,k}(\mathbb{R}^d)} \leq R\}$ there exists a unique solution $u \in C_t H_x^{k,k}([-T,T] \times \mathbb{R}^d)$ to
$$iu_t - \Delta u = \pm |u|^p u \quad \text{with} \quad u(0) = u_0.$$

Problem 7. (Failure of uniform continuity of the data-to-solution map) Let $d \geq 1$ and let $p > 0$ be an even integer such that $s_c > 0$. Show that the data-to-solution map for the initial-value problem
$$iu_t - \Delta u = \pm |u|^p u \quad \text{with} \quad u(0) = u_0$$
is not uniformly continuous in the L_x^2 topology, even on bounded sets. Specifically, show that for any $0 < \varepsilon, \delta < 1$ and any $t > 0$ there exist two solutions u_1, u_2 such that $u_1(0), u_2(0) \in \mathcal{S}(\mathbb{R}^d)$,
$$\|u_1(0)\|_{L_x^2} \leq \varepsilon, \quad \|u_1(0)\|_{L_x^2} \leq \varepsilon, \quad \text{and} \quad \|u_1(0) - u_2(0)\|_{L_x^2} \leq \delta,$$
but
$$\|u_1(t) - u_2(t)\|_{L_x^2} \gtrsim_{d,p} \varepsilon.$$
(*Hint*: Use the construction in class with zero Galilei parameter.)

Problem 8. (Energy trapping) Let W denote the unique (up to scaling) non-negative radial $\dot{H}^1(\mathbb{R}^3)$ solution to $\Delta W + W^5 = 0$. Let $u_0 \in \dot{H}^1(\mathbb{R}^3)$ be such that
$$E(u_0) \leq (1 - \delta_0) E(W) \quad \text{for some } \delta_0 > 0.$$
Prove that there exist $\delta_1, \delta_2 > 0$ such that

1) If $\|\nabla u_0\|_2^2 \leq \|\nabla W\|_2^2$, then $\|\nabla u_0\|_2^2 \leq (1 - \delta_1)\|\nabla W\|_2^2$.

2) If $\|\nabla u_0\|_2^2 \geq \|\nabla W\|_2^2$, then $\|\nabla u_0\|_2^2 \geq (1 + \delta_2)\|\nabla W\|_2^2$. In this case,
$$\int_{\mathbb{R}^3} |\nabla u_0(x)|^2 - |u_0(x)|^6 \, dx < -2(\delta_0 + \delta_2)\|\nabla W\|_2^2.$$

Here E denotes the energy functional associated to the focusing energy-critical NLS.

Problem 9. (Refined Fatou) Let $d \geq 1$, $1 \leq p < \infty$, and let $\{f_n\}_{n \geq 1} \subset L^p(\mathbb{R}^d)$ such that $\limsup \|f_n\|_{L^p} < \infty$. Show that if $f_n \to f$ almost everywhere, then
$$\int_{\mathbb{R}^d} \Big| |f_n(x)|^p - |f_n(x) - f(x)|^p - |f(x)|^p \Big| \, dx \to 0.$$