HOMEWORK 2

Problem 1. (Restriction implies Kakeya) Assume that the restriction conjecture for the sphere holds, namely,

$$\|\hat{f}\|_{L^q(S^{d-1},d\sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \quad \text{for all} \quad p < \frac{2d}{d+1} \quad \text{and} \quad \frac{d+1}{p'} \leq \frac{d-1}{q}.$$

Show that this implies the Kakeya conjecture, namely, for all $\varepsilon > 0$

$$\|f_\delta^*\|_{L^d(S^{d-1},d\sigma)} \lesssim_\varepsilon \delta^\varepsilon \|f\|_{L^d(\mathbb{R}^d)}.$$

Problem 2. a) Show that one can only obtain restriction estimates $R_K(p \to q)$ on any compact subset $K \subset S_{parab} \subset \mathbb{R}^d$ of the paraboloid if $\frac{d+1}{p'} \leq \frac{d-1}{q}$.

b) Show that one can only obtain restriction estimates $R_{S_{parab}}(p \to q)$ on the paraboloid in $\mathbb{R}^d$ if $\frac{d+1}{p'} = \frac{d-1}{q}$.

Problem 3. Fix $d \geq 1$ and $M \leq \frac{1}{4}N$. Show that

$$\big\|(e^{it\Delta} f_M)(e^{it\Delta} g_N)\big\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim M^{\frac{d-1}{2}} N^{-\frac{1}{2}} \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}.$$

Problem 4. (Existence of optimizers to the Strichartz inequality) Fix $d \geq 1$ and let

$$C_d := \sup\Big\{ \|e^{it\Delta} f\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R}\times\mathbb{R}^d)} : \|f\|_{L^2(\mathbb{R}^d)} = 1 \Big\}.$$

Show that there exists a function $\phi \in L^2(\mathbb{R}^d)$ such that

$$\|e^{it\Delta} f\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R}\times\mathbb{R}^d)} = C_d \|\phi\|_{L^2(\mathbb{R}^d)}.$$