HOMEWORK 2

Problem 1. Let $\omega : \mathbb{R}^d \to [0, \infty)$ given by $\omega(x) = |x|^\alpha$.
(a) Show that $\omega\, dx$ is a doubling measure if and only if $\alpha > -d$.
(b) Show that $\omega \in A_p$ with $1 < p < \infty$ if and only if $-d < \alpha < (p-1)d$.

Problem 2. Fix $1 \le p < \infty$ and let $\omega \in A_p$.
(a) Show that $M_\omega : L^1(\omega\, dx) \to L^{1,\infty}(\omega\, dx)$, where

$$M_\omega f(x) = \sup_{r>0} \frac{1}{\omega(B(x,r))} \int_{B(x,r)} |f(y)|\omega(y)\, dy.$$

(b) Show that $(Mf)^p \lesssim M_\omega(f^p)$ for all $f \ge 0$, where M denotes the Hardy-Littlewood maximal function.
(c) Conclude that $M : L^p(\omega\, dx) \to L^{p,\infty}(\omega\, dx)$.

Problem 3. The dyadic cubes in $\mathbb{R}^d$ are sets of the form

$$Q_{n,k} = [k_1 2^n, (k_1+1)2^n) \times \cdots \times [k_d 2^n, (k_d+1)2^n),$$

where n ranges over $\mathbb{Z}$ and $k \in \mathbb{Z}^d$.
(a) Given a collection of dyadic cubes with bounded maximal diameter, show that one may find a subcollection which covers the same region of $\mathbb{R}^d$, but with all cubes disjoint.
(b) Define the dyadic maximal function by

$$[M_D f](x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where the supremum is taken over all dyadic cubes that contain x. Show M_D is of weak-type $(1,1)$ and of type (p,p) for all $1 < p \le \infty$.

Problem 4. Let M_D denote the dyadic maximal function defined above and let $Q_0 := [0,1)^d$.
(a) For $\alpha > 0$, show that

$$|\{x \in \mathbb{R}^d : [M_D f](x) > \alpha\}| \lesssim \frac{1}{\alpha} \int_{|f| > c\alpha} |f(y)|\, dy$$

for some small constant c.
(b) Deduce that if f is supported on Q_0 and $|f| \log[2 + |f|] \in L^1(Q_0)$, then $M_D f \in L^1(Q_0)$.
(c) Given $f \in L^1(Q_0)$ and $\alpha > \int_{Q_0} |f(y)|\, dy$, show that

$$|\{x \in Q_0 : [M_D f](x) > \alpha\}| \gtrsim \frac{1}{\alpha} \int_{|f| > \alpha} |f(y)|\, dy$$

Hint: perform a Calderon–Zygmund style decomposition.
(d) Deduce that if $M_D f \in L^1(Q_0)$, then $|f| \log[2 + |f|] \in L^1(Q_0)$.

Problem 5 (Schur's test with weights)**.** Suppose $(X, d\mu)$ and $(Y, d\nu)$ are measure spaces and let $w(x, y)$ be a positive measurable function defined on $X \times Y$. Let $K(x, y) : X \times Y \to \mathbb{C}$ satisfy

$$\sup_{x \in X} \int_Y w(x,y)^{\frac{1}{p}} |K(x,y)| \, d\nu(y) = C_0 < \infty, \tag{1}$$

$$\sup_{y \in Y} \int_X w(x,y)^{-\frac{1}{p'}} |K(x,y)| \, d\mu(x) = C_1 < \infty, \tag{2}$$

for some $1 < p < \infty$. Then the operator defined by

$$Tf(x) = \int_Y K(x,y) f(y) \, d\nu(y)$$

is a bounded operator from $L^p(Y, d\nu)$ to $L^p(X, d\mu)$. In particular,

$$\|Tf\|_{L^p(X,d\mu)} \lesssim C_0^{\frac{1}{p'}} C_1^{\frac{1}{p}} \|f\|_{L^p(Y,d\nu)}.$$

Remark. This is essentially a theorem of Aronszajn. When $K \geq 0$, Gagliardo has shown that the existence of a weight $w(x, y) = a(x)b(y)$ obeying (1) and (2) is necessary for the L^p boundedness of T.

Problem 6 (Hardy's inequality)**.** Let $f \in \mathcal{S}(\mathbb{R}^d)$ and $0 \leq s < d$. Show that

$$\Big\| \frac{f(x)}{|x|^s} \Big\|_p \lesssim \| |\nabla|^s f \|_p \quad \text{for all} \quad 1 < p < \frac{d}{s}.$$

Hint: Show that there exists $g \in L^p$ so that $f = |\nabla|^{-s} g$ and then use Problem 5 for the kernel $K(x, y) = |x|^{-s} |x - y|^{s-d}$.