HOMEWORK 1

Problem 1. Prove Young's inequality: For $1 \le p, q, r \le \infty$,

$$\|f * g\|_{L^r(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)} \quad \text{whenever} \quad 1 + \tfrac{1}{r} = \tfrac{1}{p} + \tfrac{1}{q}.$$

Show that no inequalities of this type are possible for other exponents.

Problem 2. Let $\|\cdot\|$ denote a quasinorm on functions. Let $f_1, \dots f_N$ be functions satisfying the bounds

$$\|f_n\| \le 2^{-\varepsilon n}$$

for some $\varepsilon > 0$. Show that

$$\Big\|\sum_{n=1}^{N} f_n\Big\| \lesssim_\varepsilon 1,$$

where the implicit constant is independent of N.
Hint: First reduce the problem to large positive ε.

Problem 3. Let $1 \le p < \infty$ and $1 \le q \le \infty$ and let $f \in L^{p,q}(\mathbb{R}^n)$. We can write $f = \sum f_n$ where

$$f_n = f\chi_{\{x\colon H_{n+1} < |f(x)| \le H_n\}} \quad \text{with} \quad H_n = \inf\{\lambda : |\{x : |f(x)| > \lambda\}| \le 2^{n-1}\}.$$

Show that

$$\|f\|^*_{L^{p,q}} \sim \|H_n 2^{\frac{n}{p}}\|_{\ell^q(\mathbb{Z})}.$$

Hint: Show that for $H_{n+1} < \lambda \le H_n$ we have $2^{n-1} \le |\{x : |f(x)| > \lambda\}| < 2^n$.

Problem 4. Show that if

$$|f| \le \sum H_n \chi_{E_n} \quad \text{with} \quad |E_n| \lesssim 2^n,$$

then

$$\|f\|^*_{L^{p,q}} \lesssim \|H_n 2^{\frac{n}{p}}\|_{\ell^q(\mathbb{Z})}.$$

Hint: Show that $|\{x : |f(x)| > \lambda\}| \lesssim \sup\{2^n : \sum_{m \ge n} H_m > \lambda\}$.

Problem 5. (Hölder in Lorentz spaces) Let $1 \le p, p_1, p_2 < \infty$ and $1 \le q, q_1, q_2 \le \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Show that

$$\|fg\|^*_{L^{p,q}} \lesssim \|f\|^*_{L^{p_1,q_1}}\|g\|^*_{L^{p_2,q_2}}.$$

Hint: Use the previous three problems.