HOMEWORK 3

Problem 1. Let $d\sigma$ denote the surface measure on the sphere $S^{d-1} \subset \mathbb{R}^d$ for $d \geq 2$. Show that

$$|\hat{\sigma}(x)| \lesssim \langle x \rangle^{-\frac{d-1}{2}}.$$

Problem 2. (Frequency-localized dispersive estimate for the Airy propagator) Let $f \in \mathcal{S}(\mathbb{R})$. Prove that

$$\|e^{-t\partial_x^3} f_N\|_{L_x^\infty(\mathbb{R})} \lesssim \min\{|t|^{-1/3}, (N|t|)^{-1/2}\}\|f_N\|_{L_x^1(\mathbb{R})}$$

uniformly for $N \in 2^{\mathbb{Z}}$ and $t \in \mathbb{R}$.

Problem 3. Fix $f \in \mathcal{S}(\mathbb{R})$.
a) Using the result of Problem 2 and the method of TT^*, prove the following frequency-localized Strichartz estimate for the Airy propagator:

$$\||\nabla|^{1/6} e^{-t\partial_x^3} f_N\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{R})} \lesssim \|f_N\|_{L_x^2(\mathbb{R})}$$

uniformly for $N \in 2^{\mathbb{Z}}$.
b) Deduce that

$$\||\nabla|^{1/6} e^{-t\partial_x^3} f\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{R})} \lesssim \|f\|_{L_x^2(\mathbb{R})}.$$

Problem 4. (Improved Sobolev embedding) Let $f \in \mathcal{S}(\mathbb{R}^3)$. Prove that

$$\|f\|_{L_x^6} \lesssim \|\nabla f\|_{L_x^2}^{1/3} \sup_{N\in 2^{\mathbb{Z}}} \|f_N\|_{L_x^6}^{2/3}.$$