HOMEWORK 2

Problem 1. a) Show that

$$\|f_N\|_p + \|f_{\le N}\|_p \lesssim \|f\|_p \quad \text{for all} \quad 1 \le p \le \infty.$$

b) Show that for a Schwartz function f,

$$|f_N(x)| + |f_{\le N}(x)| \lesssim [Mf](x),$$

where Mf denotes the Hardy–Littlewood maximal function of f.
c) For $f \in L^p$ with $1 < p < \infty$ show that $\sum_{N\in 2^{\mathbb{Z}}} f_N$ converges in L^p and that the limit is f.
d) Show that

$$\|f_N\|_q + \|f_{\le N}\|_q \lesssim N^{\frac{d}{p}-\frac{d}{q}}\|f\|_p \quad \text{for all} \quad 1 \le p \le q \le \infty.$$

e) Show that

$$\||\nabla|^s f_N\|_p \sim N^s \|f_N\|_p \quad \text{for all} \quad s \in \mathbb{R} \quad \text{and} \quad 1 \le p \le \infty.$$

Deduce that

$$\||\nabla|^s f_{\le N}\|_p \lesssim N^s\|f\|_p \quad \text{and} \quad \|f_{\ge N}\|_p \lesssim N^{-s}\||\nabla|^s f\|_p$$

for all $s \ge 0$ and $1 \le p \le \infty$.

Remark. Using the fattened Littlewood–Paley projections $\tilde{P}_N = P_{N/2}+P_N+P_{2N}$, one can *a posteriori* strengthen the statement in part (d) above to read

$$\|f_N\|_q \lesssim N^{\frac{d}{p}-\frac{d}{q}}\|f_N\|_p \quad \text{and} \quad \|f_{\le N}\|_q \lesssim N^{\frac{d}{p}-\frac{d}{q}}\|f_{\le N}\|_p \quad \text{for all} \quad 1 \le p \le q \le \infty.$$

Problem 2. Show that for $f \in L^1(\mathbb{R}^d)$ the sum $\sum_{N\in 2^{\mathbb{Z}}} f_N$ need not converge to f in L^1. However, $f_{\le N}$ converges to f in L^1 as $N \to \infty$.

Problem 3 (Schur's test with weights)**.** Suppose $(X, d\mu)$ and $(Y, d\nu)$ are measure spaces and let $w(x, y)$ be a positive measurable function defined on $X \times Y$. Let $K(x, y) : X \times Y \to \mathbb{C}$ satisfy

$$\sup_{x\in X} \int_Y w(x,y)^{\frac{1}{p}} |K(x,y)|\, d\nu(y) = C_0 < \infty, \tag{1}$$

$$\sup_{y\in Y} \int_X w(x,y)^{-\frac{1}{p'}} |K(x,y)|\, d\mu(x) = C_1 < \infty, \tag{2}$$

for some $1 < p < \infty$. Then the operator defined by

$$Tf(x) = \int_Y K(x,y) f(y)\, d\nu(y)$$

is a bounded operator from $L^p(Y, d\nu)$ to $L^p(X, d\mu)$. In particular,

$$\|Tf\|_{L^p(X,d\mu)} \lesssim C_0^{\frac{1}{p'}} C_1^{\frac{1}{p}} \|f\|_{L^p(Y,d\nu)}.$$

Remark. This is essentially a theorem of Aronszajn. When $K \ge 0$, Gagliardo has shown that the existence of a weight $w(x, y) = a(x)b(y)$ obeying (1) and (2) is necessary for the L^p boundedness of T.

Problem 4 (Hardy's inequality)**.** Let $f \in \mathcal{S}(\mathbb{R}^d)$ and $0 \le s < d$. Show that

$$\Big\| \frac{f(x)}{|x|^s} \Big\|_p \lesssim \| |\nabla|^s f\|_p \quad \text{for all} \quad 1 < p < \frac{d}{s}.$$

Hint: Show that there exists $g \in L^p$ so that $f = |\nabla|^{-s} g$ and then use Problem 3 for the kernel $K(x,y) = |x|^{-s}|x-y|^{s-d}$.

Problem 5. Let $f \in \mathcal{S}(\mathbb{R}^d)$. Show that

$$\Big\| \frac{\partial^2 f}{\partial x_j \partial x_k} \Big\|_p \lesssim_p \|\Delta f\|_p \quad \text{for all} \quad 1 < p < \infty \quad \text{and} \quad 1 \le j,k \le d,$$

where $\Delta f = \sum_{j=1}^d \frac{\partial^2 f}{\partial x_j^2}$.

Problem 6 (Gagliardo–Nirenberg inequality)**.** Fix $d \ge 1$ and $0 < p < \infty$ for $d = 1, 2$ or $0 < p < \frac{4}{d-2}$ for $d \ge 3$. Show that for all $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\big\|f\big\|_{p+2}^{p+2} \le \big\|f\big\|_2^{p+2-\frac{pd}{2}} \big\|\nabla f\big\|_2^{\frac{pd}{2}}.$$

Problem 7 (Brezis–Wainger inequality)**.** Let $f \in \mathcal{S}(\mathbb{R}^2)$. Show that

$$\|f\|_{L^\infty} \lesssim \|f\|_{H^1} \big[1 + \log\big(\tfrac{\|f\|_{H^s}}{\|f\|_{H^1}}\big)\big]^{1/2} \quad \text{for all} \quad s > 1.$$

Recall that for $s > 0$, the Sobolev space $H^s(\mathbb{R}^d)$ is defined as the completion of $\mathcal{S}(\mathbb{R}^d)$ under the norm

$$\|f\|_{H^s} = \|\langle \nabla \rangle^s f\|_{L^2}$$

where $\langle x \rangle := (1 + |x|^2)^{1/2}$.

Problem 8. Given a Schwartz vector field $F : \mathbb{R}^3 \to \mathbb{C}^3$, define vector and scalar fields A and ϕ via

$$\hat{\phi}(\xi) = \frac{\xi \cdot \hat{F}(\xi)}{2\pi i |\xi|^2} \quad \text{and} \quad \hat{A}(\xi) = -\frac{\xi \times \hat{F}(\xi)}{2\pi i |\xi|^2}.$$

Note that ϕ and A are smooth functions, but need not be Schwartz.
(a) Show that

$$\|\phi\|_{L^q} + \|A\|_{L^q} \lesssim \|F\|_{L^p}$$

for $1 < p < q < \infty$ obeying $1 + \frac{d}{q} = \frac{d}{p}$.
(b) Show that $F = \nabla \times A + \nabla \phi$ and hence that

$$\|F\|_{L^p} \sim \|\nabla \times A\|_{L^p} + \|\nabla \phi\|_{L^p}$$

for any $1 < p < \infty$.
(c) Show that all (first-order) derivatives of all components of A are under control (not just the curl):

$$\|\partial_k A_l\|_{L^p} \lesssim \|F\|_{L^p}$$

for any $1 < p < \infty$ and any $k, l \in \{1, 2, 3\}$.

Remark. Observe that $F = \nabla \times A + \nabla \phi$ decomposes F into a divergence-free part and a curl-free part. Indeed, this (Helmholtz–Hodge) decomposition is orthogonal under the natural inner product on vector-valued functions. Note however, that the choice of A is far from unique; consider $A \mapsto A + \nabla \psi$. Our choice corresponds to the Coulomb gauge: $\nabla \cdot A = 0$.

Problem 9. Let $f \in L^\infty(\mathbb{R}^d)$ and fix $0 < \alpha < 1$. Show that f is α-Hölder continuous if and only if $\|P_{\geq N} f\|_{L^\infty} \lesssim N^{-\alpha}$ for all $N \geq 1$.

Problem 10. Let $f, g \in \mathcal{S}(\mathbb{R}^d)$ and $1 < p, q, r < \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Show that

$$\Big\| \sum_{N \in 2^{\mathbb{Z}}} f_N g_{\leq N} \Big\|_{L^p} \lesssim \|f\|_{L^q} \|g\|_{L^r}.$$