HOMEWORK 9

Due on Monday, June 1st, in class.

Exercise 1. (10 points) Let $f : [0, 1] \to \mathbb{R}$ be a function with Darboux's property such that for any $y \in \mathbb{R}$, $f^{-1}(\{y\})$ is closed. Prove that f is continuous.

Exercise 2. (10 points) Let $f, g : [a, b] \to [a, b]$ be two continuous functions such that $f \circ g = g \circ f$. Show that there exists $x_0 \in [a, b]$ such that $f(x_0) = g(x_0)$.

Exercise 3. (10 points) Show that there are no functions $f : [0, 1] \to \mathbb{R}$ that are continuous on $\mathbb{Q}$ and discontinuous on $\mathbb{R} \setminus \mathbb{Q}$.

Exercise 4. (10 points) Let $f : [0, 1] \to [0, 1]$ be a continuous function such that $f(0) = 0$ and $f(1) = 1$. Consider the sequence of functions $f_n : [0, 1] \to [0, 1]$ defined as follows:

$$f_1 = f, \quad f_{n+1} = f \circ f_n.$$

Prove that if $\{f_n\}_{n\geq 1}$ converges uniformly, then $f(x) = x$ for all $x \in [0, 1]$.

Exercise 5. (20 points) Let (X, d) be a metric space.
a) Let A be a non-empty subset of X. Show that the function $d(\cdot, A) : X \to \mathbb{R}$ which measures the distance of a point to A is uniformly continuous.
b) Let A and B be separated non-empty subsets of X. Show that

$$\{x \in X \,|\, d(x, A) < d(x, B)\} \quad \text{and} \quad \{x \in X \,|\, d(x, B) < d(x, A)\}$$

are disjoint open sets containing A and B, respectively. Show also that the set $\{x \in X \,|\, d(x, A) = d(x, B)\}$ is closed.

Exercise 6. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-x^2}$. Find
a) an open set $D \subseteq \mathbb{R}$ such that $f(D)$ is not open;
b) a closed set $F \subseteq \mathbb{R}$ such that $f(F)$ is not closed;
c) a set $A \subseteq \mathbb{R}$ such that $f(\bar{A}) \subsetneq \overline{f(A)}$.

Exercise 7. (10 points)
a) Let I be a bounded interval. Prove that if $f : I \to \mathbb{R}$ is uniformly continuous on I, then f is bounded on I.
b) Prove that (a) may be false if I is unbounded or if f is merely continuous.

Exercise 8. (10 points) Prove that a polynomial of degree n is uniformly continuous on $\mathbb{R}$ if and only if $n = 0$ or $n = 1$.

Exercise 9. (10 points) Consider $f : [0, 1] \to \mathbb{R}$,

$$f(x) = \begin{cases} x \ln \frac{1}{x}, & x > 0 \\ 0, & x = 0. \end{cases}$$

Show that $f([0, 1]) \subseteq [0, 1]$, f is continuous, but there does not exist $\alpha > 0$ such that

$$|f(x) - f(y)| \leq \alpha |x - y|, \quad \forall x, y \in [0, 1].$$