HOMEWORK 5

Due on Monday, May 4th, in class.

Exercise 1. Prove that the set of rational numbers $\mathbb{Q}$ is of the first category. Conclude that the set of irrational numbers $\mathbb{R}\backslash\mathbb{Q}$ is of the second category.

Definition 0.1. Given a sequence $\{a_n\}_{n\geq 1} \subseteq \mathbb{R}$, we define the sequence of *partial sums* by

$$s_n = \sum_{k=1}^{n} a_k, \quad \text{for all } n \geq 1.$$

If $\{s_n\}_{n\geq 1}$ converges to some $s \in \mathbb{R}$, we say that the *series* $\sum_{n=1}^{\infty} a_n$ converges. In this case, s is called *the sum of the series* and we write

$$\sum_{n=1}^{\infty} a_n = s.$$

In all the exercises below, the series are assumed to be real.

Exercise 2. Prove that if the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Exercise 3. (The Cauchy criterion) A series $\sum_{n=1}^{\infty} a_n$ converges if and only if for any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\Big|\sum_{k=n}^{n+p} a_n\Big| < \varepsilon \quad \text{for all } n \geq n_\varepsilon \text{ and } p \in \mathbb{N}.$$

Exercise 4. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two convergent series and let $\alpha \in \mathbb{R}$. Prove that the series $\sum_{n=1}^{\infty}(a_n + b_n)$ and $\sum_{n=1}^{\infty}(\alpha a_n)$ are convergent and moreover,

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty}(\alpha a_n) = \alpha \sum_{n=1}^{\infty} a_n.$$

Exercise 5. (The Abel criterion) Let $\{x_n\}_{n\geq 1}$ be a decreasing sequence of real numbers such that $\lim_{n\to\infty} x_n = 0$. Let $\{y_n\}_{n\geq 1}$ be a real sequence such that the sequence of partial sums $\{\sum_{k=1}^{n} y_k\}_{n\geq 1}$ is bounded. Prove that the series $\sum_{n=1}^{\infty} x_n y_n$ converges.
As a consequence, derive the following criterion due to Leibniz: if $\{x_n\}_{n\geq 1}$ is a decreasing sequence and $\lim_{n\to\infty} x_n = 0$, then the series $\sum_{n=1}^{\infty}(-1)^n x_n$ converges.

Exercise 6. Let $x \in \mathbb{R}$. Prove that the series $\sum_{n=1}^{\infty} x^n$ converges if and only if $|x| < 1$. Moreover, if $|x| < 1$, then

$$\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}.$$

Remark 0.2. For the remaining exercises, we will only consider series with non-negative terms, that is, $\sum_{n=1}^{\infty} a_n$ with $a_n \geq 0$. Note that in this case, the sequence of partial sums $\{s_n\}_{n\geq 1}$ is increasing and hence the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\{s_n\}_{n\geq 1}$ is bounded, that is, there exists $M > 0$ such that

$$\sum_{k=1}^{n} a_k \leq M, \quad \text{for all } n \geq 1.$$

Exercise 7. (The comparison criterion) Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be two series with non-negative terms such that there exists $n_0 \in \mathbb{N}$ with the property

$$x_n \leq y_n \quad \text{for all } n \geq n_0.$$

Prove that if the series $\sum_{n=1}^{\infty} y_n$ converges, then the series $\sum_{n=1}^{\infty} x_n$ also converges.

Exercise 8. (The ratio test) Let $\sum_{n=1}^{\infty} x_n$ be a series with non-negative terms such that there exists $n_0 \in \mathbb{N}$ and $0 \leq q < 1$ such that

$$\frac{x_{n+1}}{x_n} \leq q \quad \text{for all } n \geq n_0.$$

Prove that the series $\sum_{n=1}^{\infty} x_n$ converges.

Exercise 9. (The root test) Let $\sum_{n=1}^{\infty} x_n$ be a series with non-negative terms such that there exists $n_0 \in \mathbb{N}$ and $0 \leq q < 1$ such that

$$\sqrt[n]{x_n} \leq q \quad \text{for all } n \geq n_0.$$

Prove that the series $\sum_{n=1}^{\infty} x_n$ converges.

Exercise 10. Let $\sum_{n=1}^{\infty} x_n$ be a series with non-negative terms such that the sequence $\{x_n\}_{n\geq 1}$ is decreasing. Prove that the series $\sum_{n=1}^{\infty} x_n$ converges if and only if the series $\sum_{n=1}^{\infty} 2^n x_{2^n}$ converges.
As a consequence, derive that the series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges if and only if $\alpha > 1$.