HOMEWORK 4

Due on Monday, April 27th, in class.

Exercise 1. (10 points) Let (X, d) be a metric space. Prove that if a sequence $\{x_n\}_{n\geq 1} \subseteq X$ is convergent, then its limit is unique.

Exercise 2. (10 points) Let (X, d) be a metric space. Prove that a sequence $\{x_n\}_{n\geq 1} \subseteq X$ converges to some $x \in X$ if and only if every subsequence of $\{x_n\}_{n\geq 1}$ converges to x.

Exercise 3. (10 points) Let (X, d) be a metric space and let $\{x_n\}_{n\geq 1} \subseteq X$ be a convergent sequence. Prove that $\{x_n\}_{n\geq 1}$ is bounded, that is, there exist $a \in X$ and $r > 0$ such that $\{x_n\}_{n\geq 1} \subseteq B_r(a)$.

Exercise 4. (10 points) Let (X, d) be a metric space and let $A \subseteq X$ be complete. Show that A is closed.

Exercise 5. (10 points) Let (X, d) be a complete metric space and let $F \subseteq X$ be a closed set. Show that F is complete.

Exercise 6. (10 points) Let $1 < p < \infty$ and let

$$l^p = \{\{x_n\}_{n\geq 1} \subseteq \mathbb{R} | \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} < \infty\}.$$

Define $d_p : l^p \times l^p \to \mathbb{R}$ as follows: for any $x = \{x_n\}_{n\geq 1} \in l^p$, $y = \{y_n\}_{n\geq 1} \in l^p$,

$$d_p(x, y) = \Big(\sum_{n=1}^{\infty} |x_n - y_n|^p\Big)^{1/p}.$$

Show that (l^p, d_p) is a complete metric space.

Exercise 7. (10 points) Let

$$l^\infty = \{\{x_n\}_{n\geq 1} \subseteq \mathbb{R} | \sup_{n\geq 1} |x_n| < \infty\}.$$

Define $d_\infty : l^\infty \times l^\infty \to \mathbb{R}$ as follows: for any $x = \{x_n\}_{n\geq 1} \in l^\infty$, $y = \{y_n\}_{n\geq 1} \in l^\infty$,

$$d_\infty(x, y) = \sup_{n\geq 1} |x_n - y_n|.$$

Show that (l^∞, d_∞) is a complete metric space.

Exercise 8. (10 points) Let $\alpha > 0$ and define the sequence $\{x_n\}_{n\geq 1} \subseteq \mathbb{R}$ as follows:

$$x_1 > \sqrt{\alpha} \quad \text{and} \quad x_{n+1} = \frac{1}{2}\Big(x_n + \frac{\alpha}{x_n}\Big) \quad \text{for all } n \geq 1.$$

Prove that the sequence $\{x_n\}_{n\geq 1}$ converges to $\sqrt{\alpha}$.
Hint: prove the sequence is decreasing and bounded below.

Exercise 9. (20 points) Let $\alpha > 1$ and define the sequence $\{x_n\}_{n\geq 1} \subseteq \mathbb{R}$ as follows:

$$x_1 > \sqrt{\alpha} \quad \text{and} \quad x_{n+1} = \frac{x_n + \alpha}{x_n + 1} \quad \text{for all } n \geq 1.$$

1) Show that $\{x_{2n-1}\}_{n\geq 1}$ is decreasing and that $\{x_{2n}\}_{n\geq 1}$ is increasing.
2) Show that the sequence $\{x_n\}_{n\geq 1}$ converges to $\sqrt{\alpha}$.