HOMEWORK 2

Due on Monday, April 13th, in class.

Exercise 1. (10 points) Let $v_1 = (1, 1, -1, 0)$ and $v_2 = (1, 0, 0, 1)$ be two vectors in $\mathbb{E}^4$. Complete the set $\{v_1, v_2\}$ to an orthonormal basis for $\mathbb{E}^4$.

Exercise 2. (30 points) Let (X, d_1) and (Y, d_2) be two metric spaces. We define $d_3, d_4, d_5 : X \times Y \to \mathbb{R}$ as follows: for any $x_1, x_2 \in X$ and any $y_1, y_2 \in Y$,

$$\begin{aligned} d_3\big((x_1, y_1), (x_2, y_2)\big) &= d_1(x_1, x_2) + d_2(y_1, y_2) \\ d_4\big((x_1, y_1), (x_2, y_2)\big) &= \max\{d_1(x_1, x_2), d_2(y_1, y_2)\} \\ d_5\big((x_1, y_1), (x_2, y_2)\big) &= \sqrt{[d_1(x_1, x_2)]^2 + [d_2(y_1, y_2)]^2}. \end{aligned}$$

Prove that d_3, d_4, d_5 are all metrics on $X \times Y$.

Exercise 3. (10 points) Let (X, d) be a metric space and let A, B be two non-empty subsets of X. Prove that if $A \cap B \neq \emptyset$, then we have the following inequality for the diameters:

$$\delta(A \cup B) \leq \delta(A) + \delta(B).$$

Exercise 4. (10 points) Let X be a non-empty set and let $d : X \times X \to \mathbb{R}$ defined as follows: for any $x, y \in X$,

$$d(x, y) = \begin{cases} 0, & \text{if} \quad x = y \\ 1, & \text{if} \quad x \neq y. \end{cases}$$

Prove that (X, d) is a metric space. Find the open and the closed subsets of this metric space.

Exercise 5. (10 points) Let (X, d_1) be a metric space and let $d_2 : X \times X \to \mathbb{R}$ be the metric defined as follows: for any $x, y \in X$,

$$d_2(x, y) = \frac{d_1(x, y)}{1 + d_1(x, y)}.$$

Prove that a subset A of X is open with respect to the distance d_1 if and only if it is open with respect to the distance d_2.

Exercise 6. (10 points) Consider the two metrics on $\mathbb{R}^n$ given by

$$\begin{aligned} d_2(x, y) &= \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2} \\ d_\infty(x, y) &= \max_{1 \leq k \leq n} |x_k - y_k|. \end{aligned}$$

Prove that a set $A \subseteq \mathbb{R}^n$ is open with respect to the distance d_2 if and only if it is open with respect to the distance d_∞.

Exercise 7. (10 points) Let (X, d) be a metric space and let A be a non-empty subset of X. Prove that A is open if and only if it can be written as the union of a family of open balls of the form $B_r(x) = \{y \in X \,|\, d(x, y) < r\}$.

Exercise 8. (10 points) Fix $r > 0$. Let (X, d) be a metric space and let A be a non-empty subset of X with diameter $\delta(A) < r$. Let $a \in X$ and assume that $A \cap B_r(a) \neq \emptyset$. Then $A \subseteq B_{2r}(a)$.