HOMEWORK 9

Due on Monday, March 9th, in class.

Exercise 1. (10 points) Let V be a vector space over a field F. Then $W \subseteq V$ is a subspace if and only if for every $v, w \in W$ and every $\alpha, \beta \in F$ we have $\alpha v + \beta w \in W$.

Exercise 2. (10 points) Let V be a vector space over a field F and let W_1, W_2 be subspaces. Prove that $W_1 \cup W_2$ is a subspace if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Exercise 3. (20 points) Let V be a vector space over a field F and let W_1, W_2 be subspaces. Let
$$W_1 + W_2 = \{w_1 + w_2 |\, w_1 \in W_1,\ w_2 \in W_2\}.$$
1) Show that $W_1 + W_2$ is a subspace of V.
2) Show that
$$\max(\dim(W_1), \dim(W_2)) \leq \dim(W_1 + W_2) \leq \dim(W_1) + \dim(W_2).$$
3) Show that if $W_1 \cap W_2 = \{0\}$ then
$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2).$$
4) Show that
$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

Exercise 4. (10 points) Let $T : V \to W$ be a linear transformation between two vector spaces V, W over a field F. Prove that T is injective if and only if $\ker T = \{0\}$.

Exercise 5. (10 points) Let $T : V \to W$ be a linear transformation between two vector spaces V, W over a field F. Show that T is an isomorphism if and only if it maps any basis of V to a basis of W.

Exercise 6. (30 points) In each of the following cases, describe the image and the kernel of the linear transformation, calculate their dimensions, and write down explicit bases for each.
1) $T : \mathbb{R}^4 \to \mathbb{R}^2$ given by $T(x, y, z, w) = (x + y + z, x - y - w)$.
2) $T : \mathbb{Q}^3 \to \mathbb{Q}^3$ given by $T(x, y, z) = (x - 2y + z, x + 2y + z, y)$.
3) $T : \mathbb{R}^3 \to \mathbb{R}^4$ given by $T(x, y, z) = (x, x + y, y + z, x + z)$.

Exercise 7. (10 points) Let $T : V \to W$ be a linear transformation between two finite dimensional vector spaces V, W over a field F. Assume that $\dim V > \dim W$. Then there exists a vector $v \in V \backslash \{0\}$ with $T(v) = 0$.