HOMEWORK 8

Due on Wednesday, March 4th, in class.

Exercise 1. (10 points) Let V be a vector space over a field F. Show that a set of vectors $\{v_1, \ldots, v_n\} \subseteq V$ is a basis for V if and only if every element of V can be written as a unique linear combination of $v_1, \ldots, v_n$.

Exercise 2. (10 points) Let V be a vector space over a field F. Let $\{v_1, \ldots, v_n\} \subseteq V$ be a set of vectors spanning V. Show that if v_n is a linear combination of $v_1, \ldots, v_{n-1}$ then $\{v_1, \ldots, v_{n-1}\}$ also spans V.

Exercise 3. (10 points) Let $V = \{a_3x^3 + a_2x^2 + a_1x + a_0 |\, a_i \in \mathbb{Q} \text{ for } 0 \leq i \leq 3\}$ denote the space of all polynomial functions of degree less than four. Consider the set of vectors $S = \{x^3 - 3,\ x^2 + x,\ x^2 - x,\ x^3 - x^2 + 1\}$.
1) Show that V is a finite dimensional vector space over $\mathbb{Q}$. What is its dimension?
2) Write each of the vectors $x^3, x^2, x, 1$ as linear combinations of vectors in S.
3) Is S a basis?

Exercise 4. (10 points) In each of the following cases determine if the given set of vectors is linearly independent, if it spans the given vector space, and if it is a basis.
1) $v_1 = (9, 3, -1),\ v_2 = (2, 0, 2),\ v_3 = (-4, 0, 8)$ in $\mathbb{R}^3$.
2) $v_1 = (9, 3, 0, -1),\ v_2 = (2, 0, 0, 2),\ v_3 = (-4, 0, 1, 8)$ in $\mathbb{R}^4$.
If the vectors are independent but do not span V add vectors to obtain a basis.

Exercise 5. (60 points) In each of the following cases determine if the given space is a vector space:
1) The space $V = \mathbb{Z}$ over the field $\mathbb{Q}$.
2) The space $V = \{(x, y, z) \in \mathbb{R}^3 |\, x \geq 0\}$ over $\mathbb{R}$ with the usual addition and scalar multiplication on $\mathbb{R}^3$.
3) The space $V = \{(x, y, z) \in \mathbb{R}^3 |\, 2x + y = 0\}$ over $\mathbb{R}$ with the usual addition and scalar multiplication on $\mathbb{R}^3$.
4) The space $V = \{(x, y, z) \in \mathbb{R}^3 |\, x + 2y = 0 \ \text{ and } z = 3x\}$ over $\mathbb{R}$ with the usual addition and scalar multiplication on $\mathbb{R}^3$.
5) The space of functions $V = \{f : \mathbb{R} \to \mathbb{R} |\, f(0) = 0\}$ over $\mathbb{R}$ with addition and scalar multiplication given by

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (\alpha \cdot f)(x) = \alpha f(x), \quad \text{for all } f, g \in V,\ \alpha \in \mathbb{R}.$$

6) The space of functions $V = \{f : \mathbb{R} \to \mathbb{R} |\, f(1) = 1\}$ over $\mathbb{R}$ with the same addition and scalar multiplication as above.