HOMEWORK 6

Due on Wednesday, February 18th, in class.

Exercise 1. Prove that the limit of a convergent sequence is unique.

Exercise 2. Show that if a sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers converges to a, then the sequence $\{|a_n|\}_{n\in\mathbb{N}}$ converges to $|a|$. Show (via an example) that the converse is not true.

Exercise 3. Prove that a sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers converges to a if and only if each of its subsequences converges to a.

Exercise 4. Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be sequences of real numbers such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Prove that

$$\lim_{n\to\infty}(a_n + b_n) = a + b \quad \text{and} \quad \lim_{n\to\infty}(a_n b_n) = ab.$$

Exercise 5. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers converging to 0 and let $\{b_n\}_{n\in\mathbb{N}}$ be a bounded sequence of real numbers. Prove that the sequence $\{a_n b_n\}_{n\in\mathbb{N}}$ converges to 0.

Exercise 6. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of rational numbers defined as follows:

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = a_n + \frac{1}{3^n} \quad \text{for all} \quad n \geq 1.$$

Show that the sequence $\{a_n\}_{n\in\mathbb{N}}$ converges and find its limit.

Exercise 7. (In this exercise you will see a sequence of rational numbers converging to an irrational number.) Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence defined by the following rule:

$$a_1 = 3 \quad \text{and} \quad a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n} \quad \text{for all} \quad n \geq 1.$$

1) Show that this is a sequence of rational numbers that is bounded below and monotonically decreasing.
2) Deduce that $\{a_n\}_{n\in\mathbb{N}}$ converges and find its limit.

Exercise 8. For $n \geq 1$ let $a_n = \sqrt{n+1} - \sqrt{n}$. Prove that the sequence $\{a_n\}_{n\in\mathbb{N}}$ converges and find its limit.

Exercise 9. Let a_1, b_1 be two real numbers such that $0 < a_1 < b_1$. For $n \geq 1$, we define

$$a_{n+1} = \sqrt{a_n b_n} \quad \text{and} \quad b_{n+1} = \frac{a_n + b_n}{2}.$$

Prove that the sequences $\{a_n\}_{n\in N}$ and $\{b_n\}_{n\in N}$ converge to the same limit.

Exercise 10. Let $\mathcal{C}$ be the set of Cauchy sequences of rational numbers. Define the relation $\sim$ as follows: if $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in N} \in \mathcal{C}$, we write $\{a_n\}_{n\in\mathbb{N}} \sim \{b_n\}_{n\in\mathbb{N}}$ if the sequence $\{a_n - b_n\}_{n\in\mathbb{N}}$ converges to zero. For $\{a_n\}_{n\in\mathbb{N}} \in \mathcal{C}$, we denote its equivalence class by $[a_n]$. Let R denote the set of equivalence classes in $\mathcal{C}$.
1) Prove that $\sim$ is an equivalence relation on $\mathcal{C}$.
2) We define addition and multiplication on R as follows:

$$[a_n] + [b_n] = [a_n + b_n] \quad \text{and} \quad [a_n] \cdot [b_n] = [a_n b_n].$$

Show that these internal laws of composition are well defined and that R together with these operations is a commutative ring with 1.

3) We define a relation on R as follows: we write $[a_n] < [b_n]$ if $[a_n] \neq [b_n]$ and there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $a_n < b_n$. Prove that this relation is well defined. Show that the set of positive elements in R, that is,

$$P = \{[a_n] \in R | \, [a_n] > 0\}$$

obeys the following properties:

01') For every $[a_n] \in R$, exactly one of the following holds: either $[a_n] = [0]$ or $[a_n] \in P$ or $-[a_n] \in P$, where $[0]$ denotes the equivalence class of the sequence identically equal to zero.

02') For every $[a_n], [b_n] \in P$, we have $[a_n] + [b_n] \in P$ and $[a_n] \cdot [b_n] \in P$.

Conclude that R is an ordered ring.