HOMEWORK 3

Due on Wednesday, January 28th, in class.

Exercise 1. Show that a set with cardinality $\aleph_0$ is infinite.

Exercise 2. Suppose that A, B, and C are subsets of a set X such that $A \subseteq B \subseteq C$. Show that if A and C have the same cardinality, then A and B have the same cardinality.

Exercise 3. Show that $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by

$$g(n, m) = \frac{(n + m - 2)(n + m - 1)}{2} + n \quad \text{for all} \quad n, m \in \mathbb{N}$$

is bijective. Conclude that the cardinality of $\mathbb{N} \times \mathbb{N}$ is $\aleph_0$.

Exercise 4. Fix $n \geq 1$. Show that if $A_1, A_2, \ldots, A_n$ are countable, then $A_1 \times A_2 \times \ldots \times A_n$ is countable.

Exercise 5. Prove that a subset of a countable set is countable.

Exercise 6. Show that the set of rational numbers $\mathbb{Q}$ is countably infinite.

Exercise 7. Show that a countable union of countable sets is countable.

Exercise 8. Show that the set of all polynomial functions with integer coefficients is a countable set. Conclude that the set of *algebraic numbers* (that is, real numbers that are roots of polynomials with integer coefficients) is countable.

Exercise 9. Show that the cardinality of $\mathbb{R}$ is $2^{\aleph_0}$. You may use the fact that the interval $(0, 1)$ has cardinality $2^{\aleph_0}$.

Exercise 10. Prove that the set of irrational numbers has the cardinality of $\mathbb{R}$.