HOMEWORK 1

Due on Wednesday, January 14th, in class.

Exercise 1. Using the truth table show that $A \Rightarrow B$ is logically equivalent to (not B) $\Rightarrow$ (not A).

Exercise 2. Negate the following sentences:
• If pigs had wings, they would fly.
• If that plane leaves and you are not on it, then you will regret it.
• For everyone of us there is someone to make us unhappy.
• For every problem there is a solution that is neat, plausible, and wrong.

Exercise 3. Let X and Y be statements. If we want to DISPROVE the claim that "At least one of X and Y are true", which one of the following do we need to show?
a) At least one of X and Y are false.
b) X and Y are both false.
c) Exactly one of X and Y are false.
d) Y is false.
e) X does not imply Y, and Y does not imply X.
f) X is true if and only if Y is false.
g) X is false.

Exercise 4. Let $P(x)$ be a property about some object x of type X. If we want to DISPROVE the claim that "$P(x)$ is true for all x of type X", which one of the following do we have to do?
a) Show that for every x in X, $P(x)$ is false.
b) Show that for every x in X, there is a y not equal to x for which $P(y)$ is true.
c) Show that $P(x)$ being true does not necessarily imply that x is of type X.
d) Show that there are no objects x of type X.
e) Show that there exists an x of type X for which $P(x)$ is false.
f) Show that there exists an x which is not of type X, but for which $P(x)$ is still true.
g) Assume there exists an x of type X for which $P(x)$ is true, and derive a contradiction.

Exercise 5. Let X,Y,Z be statements. Suppose we know that "X is true implies Y is true", and "X is false implies Z is true". If we know that Z is false, then which one of the following can we conclude?
a) X is false.
b) X is true.
c) Y is true.
d) b) and c).
e) a) and c).
f) a), b), and c).
g) None of the above conclusions can be drawn.

Exercise 6. Let $P(n, m)$ be a property about two integers n and m. If we want to DISPROVE the claim that "There exists an integer n such that $P(n, m)$ is true for all integers m", then which one of the following do we need to prove?

a) If $P(n, m)$ is true, then n and m are not integers.
b) For every integer n, there exists an integer m such that $P(n, m)$ is false.
c) For every integer n, and every integer m, the property $P(n, m)$ is false.
d) For every integer m, there exists an integer n such that $P(n, m)$ is false.
e) There exists an integer n such that $P(n, m)$ is false for all integers m.
f) There exists integers n, m such that $P(n, m)$ is false.
g) There exists an integer m such that $P(n, m)$ is false for all integers n.

Exercise 7. Let X and Y be statements. If we know that X implies Y, which one of the following can we conclude?
a) X cannot be false.
b) X is true, and Y is also true.
c) If Y is false, then X is false.
d) Y cannot be false.
e) If X is false, then Y is false.
f) If Y is true, then X is true.
g) At least one of X and Y is true.

Exercise 8. Prove (i) through (x) from exercise 1.3.9 in the textbook.

Exercise 9. For $x, y \in \mathbb{R}$ define $x \sim y$ to mean that $x - y \in \mathbb{Z}$. Show that $\sim$ is an equivalence relation.

Exercise 10. Let $\mathcal{R} = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} |\ b \neq 0\}$ and let $\sim$ be the relation defined by $(a, b) \sim (c, d)$ if and only if $ad = bc$. Show that $\sim$ is an equivalence relation.

Exercise 11. Let $\mathcal{R} = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} |\ a|b\}$. Recall that for $a, b \in \mathbb{Z}$ we say that a divides b (and write $a|b$) if and only if there exists $n \in \mathbb{Z}$ such that $b = na$. Is $\mathcal{R}$ an order relation? Is it a partial order relation? What if it was a relation on the positive integers?

Exercise 12. Prove the following statement by induction: for all $n \geq 1$,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$