131A Final Name:______________________________

Problem 1. (10 points) Let $a \in \mathbb{R}$ be such that $a > -1$. Show that

$$(1+a)^n \geq 1 + na \quad \text{for all} \quad n \in \mathbb{N}.$$

Problem 2. (10 points) Prove that an increasing bounded sequence converges.

Problem 3. (10 points) Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(x) = 0$ for all **rational** numbers $x \in [0,1]$. Show that $f(x) = 0$ for all $x \in [0,1]$.

Problem 4. (10 points)
• (5 points) Let I be an open interval and let $f : I \to \mathbb{R}$ be a function. Define what it means for f to be *differentiable* at a point $a \in I$.
• (5 points) Use this definition to prove that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is differentiable on $\mathbb{R}$.

Problem 5. (10 points) Let $f : (0, 1) \to \mathbb{R}$ be a differentiable function such that $|f'(x)| \leq 1$ for all $x \in (0, 1)$. Show that f is uniformly continuous on $(0, 1)$.

Problem 6. (15 points)

- (5 points) Show that

$$\frac{2}{\pi} < \frac{\sin(x)}{x} \quad \text{for all} \quad x \in \left(0, \frac{\pi}{2}\right).$$

- (10 points) Prove that the series

$$\sum_{n\geq 1} (-1)^n \sin\left(\frac{1}{n}\right)$$

converges, but does not converge absolutely.

Problem 7. (25 points)

- (8 points) Let A and B be bounded subsets of $\mathbb{R}$. Show that

$$\sup A - \inf B = \sup\{a - b : \, a \in A, \, b \in B\}.$$

- (10 points) Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $M > 0$ be such that $|f(x)| \leq M$ for all $x \in [a, b]$. Show that for all partitions P of $[a, b]$ we have

$$U(f^2, P) - L(f^2, P) \leq 2M\big[U(f, P) - L(f, P)\big],$$

where U, L denote the upper and lower Darboux sums, respectively. (*Hint*: use the first part of this problem.)

- (3 points) Define what it means for a function to be (Darboux) *integrable.*
- (4 points) Prove that if a function $f : [a, b] \to \mathbb{R}$ is integrable, then f^2 is also integrable.

Problem 8. (10 points) Let $f : [a, b] \to \mathbb{R}$ be an integrable function such that for all $x \in [a, b]$ we have

$$\int_a^x f(t)\, dt = 0.$$

Show that if f is continuous at $x \in [a, b]$, then $f(x) = 0$.

Scratch Paper