HOMEWORK 5

Solve exercises 12.4, 12.8, 12.9(b), 12.10, 12.13, and 12.14 from the textbook.

Exercise 1. Let $\alpha > 1$ and define the sequence $\{x_n\}_{n\geq 1}$ of real numbers as follows:

$$x_1 > \sqrt{\alpha} \quad \text{and} \quad x_{n+1} = \frac{x_n + \alpha}{x_n + 1} \quad \text{for all} \quad n \geq 1.$$

1) Show that $\{x_{2n-1}\}_{n\geq 1}$ is decreasing and bounded below by $\sqrt{\alpha}$.
2) Show that $\{x_{2n}\}_{n\geq 1}$ is increasing and bounded above by $\sqrt{\alpha}$.
3) Show that the sequence $\{x_n\}_{n\geq 1}$ converges to $\sqrt{\alpha}$.

Exercise 2. Let $\{a_n\}_{n\geq 1}$ be a Cauchy sequence of real numbers. Show that $\{a_n^2\}_{n\geq 1}$ is also a Cauchy sequence.

Exercise 3. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers that is bounded above. Prove that $L = \limsup a_n$ has the following properties:

(i) For every $\varepsilon > 0$ there are only finitely many n for which $a_n > L + \varepsilon$

(ii) For every $\varepsilon > 0$ there are infinitely many n for which $a_n > L - \varepsilon$.

Exercise 4. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Prove that there can be at most one real number L with the following two properties:

(i) For every $\varepsilon > 0$ there are only finitely many n for which $a_n > L + \varepsilon$

(ii) For every $\varepsilon > 0$ there are infinitely many n for which $a_n > L - \varepsilon$.