HOMEWORK 4

Exercise 1. Solve exercises 10.7, 10.9, 10.10, and 10.12 from the textbook.

Exercise 2. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of rational numbers defined as follows:

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = a_n + \frac{1}{3^n} \quad \text{for all} \quad n \geq 1.$$

1) Show that $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and hence convergent.
2) Find its limit.

Exercise 3. (In this exercise you will see a Cauchy sequence of rational numbers converging to an irrational number.) Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence defined by the following rule:

$$a_1 = 3 \quad \text{and} \quad a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n} \quad \text{for all} \quad n \geq 1.$$

1) Show that the sequence is bounded below.
2) Show that this is a sequence of rational numbers.
3) Prove that the sequence is monotonically decreasing.
4) Deduce that $\{a_n\}_{n\in\mathbb{N}}$ converges and find its limit.

Exercise 4. Let a_1, b_1 be two real numbers such that $0 < a_1 < b_1$. For $n \geq 1$, we define

$$a_{n+1} = \sqrt{a_n b_n} \quad \text{and} \quad b_{n+1} = \frac{a_n + b_n}{2}.$$

1) Prove that the sequence $\{a_n\}_{n\in N}$ is monotonically increasing and that the sequence $\{b_n\}_{n\in N}$ is monotonically decreasing.
2) Show that the sequences $\{a_n\}_{n\in N}$ and $\{b_n\}_{n\in N}$ are bounded.
3) Deduce that the two sequences converge and prove that they converge to the same limit.