# HOMEWORK 4

Due on Friday, October 21st, in class.

**Exercise 1.** (40 points) Solve exercises 10.7, 10.9, 10.10, and 10.12 from the textbook.

**Exercise 2.** (20 points) Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of rational numbers defined as follows:

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = a_n + \frac{1}{3^n} \quad \text{for all} \quad n \geq 1.$$

1) Show that $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and hence convergent.
2) Find its limit.

**Exercise 3.** (20 points) (In this exercise you will see a Cauchy sequence of rational numbers converging to an irrational number.) Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence defined by the following rule:

$$a_1 = 3 \quad \text{and} \quad a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n} \quad \text{for all} \quad n \geq 1.$$

1) Show that the sequence is bounded below.
2) Show that this is a sequence of rational numbers.
3) Prove that the sequence is monotonically decreasing.
4) Deduce that $\{a_n\}_{n\in\mathbb{N}}$ converges and find its limit.

**Exercise 4.** (20 points) Consider the following sequence:

$$a_1 = \sqrt{2} \quad \text{and} \quad a_{n+1} = \sqrt{2 + a_n} \quad \text{for all } n \geq 1.$$

1) Show that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is bounded above.
2) Prove that the sequence is monotonically increasing.
3) Deduce that $\{a_n\}_{n\in\mathbb{N}}$ converges and find its limit.