HOMEWORK 1

Due on Friday, September 30th, in class.

Exercise 1. (5 points) Using the truth table show that $A \Rightarrow B$ is logically equivalent to (not B) $\Rightarrow$ (not A).

Exercise 2. (15 points) Negate the following sentences:
- If pigs had wings, they would fly.
- If that plane leaves and you are not on it, then you will regret it.
- For every problem there is a solution that is neat, plausible, and wrong.

Exercise 3. (4 points) Let X and Y be statements. If we want to DISPROVE the claim that "At least one of X and Y are true", which one of the following do we need to show?
a) At least one of X and Y are false.
b) X and Y are both false.
c) Exactly one of X and Y are false.
d) Y is false.
e) X does not imply Y, and Y does not imply X.
f) X is true if and only if Y is false.
g) X is false.

Exercise 4. (4 points) Let X and Y be statements. If we know that X implies Y, which one of the following can we conclude?
a) X cannot be false.
b) X is true and Y is also true.
c) If Y is false, then X is false.
d) Y cannot be false.
e) If X is false, then Y is false.
f) If Y is true, then X is true.
g) At least one of X and Y is true.

Exercise 5. (4 points) Let X,Y,Z be statements. Suppose we know that "X is true implies Y is true", and "X is false implies Z is true". If we know that Z is false, then which one of the following can we conclude?
a) X is false.
b) X is true.
c) Y is true.
d) b) and c).
e) a) and c).
f) a), b), and c).
g) None of the above conclusions can be drawn.

Exercise 6. (4 points) Let X and Y be statements. Which of the following strategies is NOT a valid way to show that "X implies Y"?
a) Assume that X is true and then use this to show that Y is true.
b) Show that X implies some intermediate statement Z and then show that Z implies Y.

c) Assume that X is true, and Y is false, and deduce a contradiction.
d) Assume that Y is false and then use this to show that X is false.
e) Show that some intermediate statement Z implies Y and then show that X implies Z.
f) Assume that X is false, and Y is true, and deduce a contradiction.
g) Show that either X is false, or Y is true, or both.

Exercise 7. (4 points) Let $P(x)$ be a property about some object x of type X. If we want to DISPROVE the claim that "$P(x)$ is true for all x of type X", which one of the following do we have to do?
a) Show that for every x in X, $P(x)$ is false.
b) Show that for every x in X, there is a y not equal to x for which $P(y)$ is true.
c) Show that $P(x)$ being true does not necessarily imply that x is of type X.
d) Show that there are no objects x of type X.
e) Show that there exists an x of type X for which $P(x)$ is false.
f) Show that there exists an x which is not of type X, but for which $P(x)$ is still true.
g) Assume there exists an x of type X for which $P(x)$ is true, and derive a contradiction.

Exercise 8. (4 points) Let $P(n, m)$ be a property about two integers n and m. If we want to DISPROVE the claim that "There exists an integer n such that $P(n, m)$ is true for all integers m", then which one of the following do we need to prove?
a) If $P(n, m)$ is true, then n and m are not integers.
b) For every integer n, there exists an integer m such that $P(n, m)$ is false.
c) For every integer n, and every integer m, the property $P(n, m)$ is false.
d) For every integer m, there exists an integer n such that $P(n, m)$ is false.
e) There exists an integer n such that $P(n, m)$ is false for all integers m.
f) There exists integers n, m such that $P(n, m)$ is false.
g) There exists an integer m such that $P(n, m)$ is false for all integers n.

Exercise 9. (10 points) Solve exercise 1.8 from the textbook.

Exercise 10. (10 points) Solve exercise 1.11 from the textbook.

Exercise 11. (8 points) Prove that

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

for all natural numbers n.

Exercise 12. (8 points) Prove that $2^{n+2} > 2n + 5$ for all natural numbers n.

Exercise 13. (10 points) Decide for which natural numbers the inequality $3^n > n^3$ is true. Prove your claim using mathematical induction.

Exercise 14. (10 points) Prove that $n^5 - n$ is divisible by 30 for all natural numbers n.