HOMEWORK 9

Due on Monday, November 30th, in class.

Exercise 1 (10 points)**.** *Let $f, g : [a, b] \to \mathbb{R}$ be two functions continuous on $[a, b]$ and differentiable on (a, b). Then there exists $x \in (a, b)$ such that*

$$f'(x)\big[g(b) - g(a)\big] = g'(x)\big[f(b) - f(a)\big].$$

Exercise 2 (L'Hospital's Rule, 10 points)**.** *Let $f, g : (a, b) \to \mathbb{R}$ be differentiable and such that $g'(x) \neq 0$ for all $x \in (a, b)$. Assume*

$$\lim_{x \to a+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}. \tag{1}$$

If

$$\lim_{x \to a+} f(x) = \lim_{x \to a+} g(x) = 0, \tag{2}$$

then

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = L. \tag{3}$$

Remark 1. *Note that*

- *The hypothesis* (2) *can be replaced by* $\lim_{x \to a+} |g(x)| = \infty$.
- *The limits in* (1), (2), *and* (3) *can be replaced by* $\lim_{x \to a-}$, $\lim_{x \to a}$, $\lim_{x \to \infty}$, *or* $\lim_{x \to -\infty}$, *provided that the interval on which f, g are defined is modified accordingly.*
- *L can be taken to be ∞ or $-\infty$.*

Exercise 3 (20 points)**.** *Solve exercises 30.1, 30.2, 30.3, and 30.5 from the textbook.*

Definition 1. *Let I be an open interval containing $x = 0$ and let $f : I \to \mathbb{R}$ be a function that has derivatives of all orders at $x = 0$. Then the series*

$$\sum_{k \geq 0} \frac{f^{(k)}(0)}{k!} \, x^k$$

is called the Taylor series for f about 0. The remainder $R_n(x)$ *is defined by*

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} \, x^k$$

Note that $f(x) = \sum_{k \geq 0} \frac{f^{(k)}(0)}{k!} \, x^k$ *if and only if* $\lim_{n \to \infty} R_n(x) = 0$.

Exercise 4 (Taylor's formula, 10 points)**.** *Let $a < 0 < b$ and $f : (a, b) \to \mathbb{R}$. Assume the n^{th} derivative of f exists on (a, b). Then for all $x \in (a, b) \backslash \{0\}$ there exists y between 0 and x such that*

$$R_n(x) = \frac{f^{(n)}(y)}{n!} \, x^n.$$

Exercise 5 (10 points)**.** *Let* $a < 0 < b$ *and* $f : (a, b) \to \mathbb{R}$*. If all derivatives of* f *exist on* (a, b) *and are bounded by some* $M > 0$*, then*

$$\lim_{n\to\infty} R_n(x) = 0 \quad \textit{for all} \quad x \in (a, b).$$

Exercise 6 (40 points)**.** *Solve exercises 32.3, 32.6, 32.7, and 32.8 from the textbook.*