HOMEWORK 4

Due on Friday, October 23rd, in class.

Exercise 1. (40 points) Solve exercises 10.8, 10.9, 10.10, and 10.12 from the textbook.

Exercise 2. (15 points) (In this exercise you will see a sequence of rational numbers converging to an irrational number.) Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence defined by the following rule:

$$a_1 = 3 \quad \text{and} \quad a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n} \quad \text{for all} \quad n \geq 1.$$

1) Show that this is a sequence of rational numbers that is bounded below and monotonically decreasing.
2) Deduce that $\{a_n\}_{n\in\mathbb{N}}$ converges and find its limit.

Exercise 3. (15 points) Consider the following sequence:

$$a_1 = \sqrt{2} \quad \text{and} \quad a_{n+1} = \sqrt{2 + a_n} \quad \text{for all } n \geq 1.$$

1) Show that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is bounded above and monotonically increasing.
2) Deduce that $\{a_n\}_{n\in\mathbb{N}}$ converges and find its limit.

Exercise 4. (10 points) Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of rational numbers defined as follows:

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = a_n + \frac{1}{3^n} \quad \text{for all} \quad n \geq 1.$$

Show that the sequence $\{a_n\}_{n\in\mathbb{N}}$ converges and find its limit.

Exercise 5. (20 points) Let a_1, b_1 be two real numbers such that $0 < a_1 < b_1$. For $n \geq 1$, we define

$$a_{n+1} = \sqrt{a_n b_n} \quad \text{and} \quad b_{n+1} = \frac{a_n + b_n}{2}.$$

Prove that the sequences $\{a_n\}_{n\in N}$ and $\{b_n\}_{n\in N}$ converge to the same limit.