Conjugation Theorems via Neumann Series

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Abstract

Consider an analytic map of a domain in the complex plane to itself. Three basic theorems asserting the existence of an analytic conjugation to a normal form near a fixed point are due respectively to Koenigs, Leau and Boettcher. The conjugating functions are solutions to certain functional equations. We convert each of the three functional equations to a resolvent equation for a composition operator. This leads to proofs of each of the conjugation theorems as a consequence of the convergence of the Neumann series for the resolvent equation.

Let $\varphi(z)$ be an analytic map defined on a neighborhood of $z = 0$ with fixed point at 0. Then

$$\varphi(z) = \mu z + \mathcal{O}(z^2), \qquad |z| < \delta, \tag{1}$$

where $\mu = \varphi'(0)$ is called the *multiplier of* $\varphi(z)$ *at* 0. The fixed point is *attracting* if $|\mu| < 1$, *neutral* if $|\mu| = 1$, and *repelling* if $|\mu| > 1$. If $\mu = 0$ we say it is *superattracting.*

Under a change of variables $w = f(z)$, with $f(0) = 0$, the map $\varphi(z)$ is transformed to the map $\psi = f \circ \varphi \circ f^{-1}$. The map $\psi(w)$ also has a fixed point at 0, with the same multiplier as φ. The basic identity relating $\varphi(z)$ and $\psi(w)$ is given by the functional equation

$$f(\varphi(z)) = \psi(f(z)), \qquad |z| < \delta. \tag{2}$$

We say that f *conjugates* φ *to* ψ.

The idea of an analytic conjugation was introduced by E. Schröder in the early 1870's. He was interested in conjugating $\varphi(z)$ to the dilation $\psi(w) = \mu w$, in which case the basic

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conjugation identity (2) becomes *Schröder's functional equation*

$$f(\varphi(z)) = \mu f(z), \qquad |z| < \delta. \tag{3}$$

Schröder was interested in studying the iterates of $\varphi(z)$. Any f conjugating $\varphi(z)$ to $\psi(w) = \mu w$ also conjugates the mth iterate $\varphi^m(z) = (\varphi \circ \ldots \circ \varphi)(z)$ (m times) to the mth iterate $\psi^m(w) = \mu^m w$ of ψ. This makes transparent the behavior of the iterates of φ, which was Schröder's goal. The fundamental existence theorem for solutions to Schröder's equation was proved a decade later by G. Koenigs.

Theorem (Koenigs, 1884). *If the fixed point $z = 0$ for $\varphi(z)$ is attracting but not superattracting, or if it is repelling, then Schröder's equation* (3) *has an analytic solution of the form*

$$f(z) = z + \mathcal{O}(z^2), \qquad |z| < \delta.$$

Any other solution of Schröder's equation (with μ the multiplier of $\varphi(z)$ at $z = 0$) is a constant multiple of $f(z)$.

In the case of a superattracting fixed point, we can assume after a preliminary dilation that $\varphi(z)$ has the form

$$\varphi(z) = z^p + \mathcal{O}(z^{p+1}), \qquad |z| < \delta. \tag{4}$$

The power $p \geq 2$ is a conjugation invariant. In this case we might hope that $\varphi(z)$ is conjugate to the power function $\psi(w) = w^p$. The basic conjugation identity (2) becomes

$$f(\varphi(z)) = f(z)^p, \qquad |z| < \delta, \tag{5}$$

which we refer to as *Boettcher's equation.*

Theorem (Boettcher, 1904). *If $\varphi(z)$ has a superattracting fixed point at $z = 0$, then Boettcher's equation* (5) *has an analytic solution of the form*

$$f(z) = z + \mathcal{O}(z^2), \qquad |z| < \delta. \tag{6}$$

Any other solution of Boettcher's equation that is not identically zero is a multiple of $f(z)$ by a $(p-1)$th root of unity.

The case of a neutral fixed point at 0 breaks into two subcases, depending on whether the multiplier μ is a root of unity or not. The fixed point is *rationally neutral* if $\mu^m = 1$ for some integer $m \geq 1$, otherwise it is *irrationally neutral.* The rationally neutral case $\mu^m = 1$ can be reduced to the case $\mu = 1$ by replacing $\varphi(z)$ by its mth iterate $\varphi^m(z)$. We assume then that $\mu = 1$, and we also assume that $\varphi(z)$ is not the identity function z. Then after a preliminary dilation, $\varphi(z)$ has the form

$$\varphi(z) = z - z^{q+1} + \mathcal{O}(z^{q+2}) = z(1 - z^q) + \mathcal{O}(z^{q+2}),$$

for some integer $q \geq 1$. The *attracting directions* are the angles of the rays for which $z^q > 0$. Near the origin, points on these rays are mapped closer to the origin by $\varphi(z)$. The *repelling directions* are the angles of the rays for which $z^q < 0$. Points on these rays near the origin are mapped further from the origin by $\varphi(z)$.

The positive real axis corresponds to an attracting direction, and the angles $\pm\pi/q$ are repelling directions. If the sector $\{|\arg z| < \pi/q\}$ is mapped to the slit ζ-plane $\mathbb{C}\backslash(-\infty, 0]$ by $q\zeta = 1/z^q$, the map $\varphi(z)$ is conjugated to a map of the form

$$\phi(\zeta) = \zeta + 1 + \mathcal{O}(|\zeta|^{-1/q}), \qquad \zeta \in \mathbb{C}\backslash(-\infty, 0],$$

which behaves asymptotically like the translation $\psi(w) = w + 1$. In this case the goal is to conjugate $\phi(\zeta)$ to the translation $\psi(w) = w + 1$ on a sector $\{|\arg w| < \pi - \varepsilon\}$, thereby conjugating $\varphi(z)$ to $\psi(w)$ in some sector-like domain with vertex at 0. The basic conjugation identity (2) becomes *Abel's equation*

$$f(\phi(\zeta)) = f(\zeta) + 1, \qquad |\arg \zeta| < \pi - \varepsilon. \tag{7}$$

The existence of a solution of Abel's equation was first obtained by L. Leau. A more transparent existence proof together with asymptotic estimates for the solution were given some years later by P. Fatou.

Theorem (Leau, 1897; Fatou, 1919). *Let $\gamma > 0$, let $0 < \alpha < \pi$, and let $C_0 > 0$. Suppose that $\phi(\zeta)$ is an analytic map of the sector $C_0 + \{|\arg \zeta| < \alpha\}$ into itself, such that*

$$\phi(\zeta) = \zeta + 1 + \mathcal{O}(|\zeta|^{-\gamma}), \qquad |\zeta| \to \infty,\ \zeta \in C_0. \tag{8}$$

Then for any $\varepsilon > 0$ and for suitably large $C_1 > C_0$, Abel's equation (7) has an analytic solution $f(\zeta)$ in the smaller sector $C_1 + \{|\arg \zeta| < \alpha - \varepsilon\}$ satisfying

$$f(\zeta) = \begin{cases} \zeta + \mathcal{O}(\log|\zeta|), & \gamma = 1, \\ \zeta + \mathcal{O}(|\zeta|^{1-\gamma}), & \gamma \neq 1. \end{cases} \tag{9}$$

There are many proofs of these three fundamental theorems on conjugating to normal forms. Most proofs involve some sort of iteration scheme towards a fixed point in function space. Our aim is to give reasonably simple proofs of each of these theorems based in principle on the Neumann series

$$(\lambda I - T)^{-1} = \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}}, \qquad |\lambda| > \|T\|, \tag{10}$$

for a bounded linear operator T on some Banach space. The connection with the iteration schemes is as follows. For a fixed vector y, the solution of the resolvent equation $(\lambda I - T)x = y$ is the fixed point for the nonlinear mapping

$$S(x) = (y + Tx)/\lambda.$$

If $|\lambda| > \|T\|$, the mapping S is a contraction mapping. When we apply the iterates of S to the initial vector 0, we obtain precisely the partial sums of the Neumann series,

$$S^m(0) = \sum_{k=0}^{m} \frac{T^k x}{\lambda^k}.$$

The contraction mapping principle can be invoked to conclude that these iterates converge to the solution of the resolvent equation.

For the proofs of the theorems of Boettcher and Koenigs, we consider the composition operator

$$(Tf)(z) = f(\varphi(z)).$$

The Banach space we consider is the space $A(\Delta_\delta)$ of continuous function on the disk $\Delta_\delta = \{|z| \le \delta\}$ that are analytic on the interior of the disk. At this stage we are interested only in finding solutions analytic in some small neighborhood of 0, so we are allowed to shrink δ when convenient. We choose $\delta > 0$ such that $|\varphi(z)| < \delta$ when $|z| \le \delta$. Then $\varphi(\Delta_\delta)$ is a compact subset of the interior of Δ_δ, and the composition operator T is well-defined and satisfies $\|T\| \le 1$, $T(1) = 1$. Let A_0 be the subspace of functions $f \in A(\Delta_\delta)$ satisfying $f(0) = 0$, and denote by T_0 the restriction of T to A_0.

Proof of Boettcher's Theorem. In this case, $\varphi(z) = z^p + \mathcal{O}(z^{p+1})$, where $p \ge 2$. We assume that $0 < \delta < 1/2$, and that $|\varphi(z)| \le 2|z|^p$ for $|z| \le \delta$. Then $|\varphi(\varphi(z))| \le 2|\varphi(z)|^p \le 2^{1+p}|z|^{p^2}$, and proceeding by induction we obtain

$$|\varphi^m(z)| \le 2^{1+p+\cdots+p^{m-1}}|z|^{p^m} \le (2|z|)^{p^m} \le (2\delta)^{p^m}.$$

If $g \in A_0$ satisfies $\|g\| \leq 1$, then using the Schwarz lemma, we obtain

$$|(T_0^m g)(z)| = |g(\varphi^m(z))| \leq |\varphi^m(z)| \leq (2\delta)^{p^m}.$$

Consequently $\|T_0^m\| \leq (2\delta)^{p^m}$. This estimate shows that the Neumann series (10) for T_0 converges for all $\lambda \neq 0$. (Since $\|T_0^m\|^{1/m} \to 0$, the spectral radius of T_0 is zero, and T_0 is quasi-nilpotent.) We seek a solution $f(z)$ to Boettcher's equation (5), normalized so that $f(0) = 0$ and $f'(0) = 1$. Define $g, h \in A_0$ by

$$g(z) = \log(f(z)/z), \qquad h(z) = \log\left(\varphi(z)/z^p\right).$$

In terms of $g(z)$, Boettcher's equation becomes

$$g(\varphi(z)) = \log\left(\frac{f(\varphi(z))}{\varphi(z)}\right) = \log\left(\frac{f(z)^p}{\varphi(z)}\right) = \log\left(\frac{f(z)^p}{z^p}\right) - h(z) = p\, g(z) - h(z).$$

Thus Boettcher's equation is equivalent to the resolvent equation

$$(pI - T_0)g = h.$$

To prove Boettcher's theorem, we simply solve this resolvent equation for $g \in A_0$ and we set $f(z) = ze^{g(z)}$. □

Proof of Koenigs' Theorem. We assume that $\varphi(z) = \mu z + \mathcal{O}(z^2)$, where $0 < |\mu| < 1$. Choose $\varepsilon > 0$ so small that $|\mu| + \varepsilon < 1$, and choose $0 < \delta < 1$ so that

$$|\varphi(z)| \leq (|\lambda| + \varepsilon)|z|, \qquad |z| \leq \delta.$$

Then φ maps the disk $\{|z| \leq \delta\}$ into the proper subdisk $\{|z| \leq (|\lambda| + \varepsilon)\delta\}$. Let A_1 be the subspace of functions $g \in A(\Delta_\delta)$ such that $g(0) = g'(0) = 0$, and let T_1 be the restriction of T to A_1. If $g \in A_1$ satisfies $\|g\| \leq 1$, then $|g(z)| \leq |z|^2/\delta^2$, and consequently

$$|g(\varphi(z))| \leq \frac{|\varphi(z)|^2}{\delta^2} \leq \frac{(|\mu| + \varepsilon)^2|z|^2}{\delta^2} \leq (|\mu| + \varepsilon)^2.$$

Hence $\|T_1 g\| \leq (|\mu| + \varepsilon)^2$, and we conclude that

$$\|T_1\| \leq (|\mu| + \varepsilon)^2. \tag{11}$$

We may assume that ε was originally chosen so small that

$$(|\mu| + \varepsilon)^2 < |\mu|.$$

Then from (11) we have $\|T_1\| < |\mu|$, and $\mu I - T_1$ is invertible on A_1. Set $h_0(z) = z$. Then $Th_0 = \varphi(z) = \mu z + \mathcal{O}(z^2)$, so $(\mu I - T)h_0 \in A_1$. Since $\|T_1\| < |\mu|$, we can solve the resolvent equation

$$(\mu I - T_1)h_1 = (\mu I - T)h_0.$$

for $h_1 \in A_1$. If $f = h_0 - h_1 = z + \mathcal{O}(z^2)$, then $(\mu I - T)f = 0$, and f satisfies Schröder's equation (3). $\square$

The proof of Koenigs' theorem can be modified to identify the eigenvalues and eigenfunctions of T. The solution to Schröder's equation given by Koenigs' theorem is called the *principal eigenfunction* of the composition operator T. Its eigenvalue is the multiplier $\mu = \varphi'(0)$.

Theorem. *Suppose $\varphi(z) = \mu z + \mathcal{O}(z^2)$, where $0 < |\mu| < 1$, and suppose $f(z)$ is the principal eigenfunction of the composition operator T. Suppose $g(z)$ is analytic in some neighborhood of 0 and is not identically zero. If $g(z)$ and λ satisfy*

$$g(\varphi(z)) = \lambda g(z)$$

for z near 0, then $\lambda = \mu^m$ for some integer $m \geq 0$, and $g(z)$ is a constant multiple of $f(z)^m$.

Proof. First note that the function $f_m(z) = f(z)^m = z^m + \mathcal{O}(z^{m+1})$ is an eigenfunction of T with eigenvalue μ^m,

$$(Tf_m)(z) = f(\varphi(z))^m = \mu^m f(z)^m = \mu^m f_m(z).$$

Fix $m \geq 1$. We proceed as above, choosing ε so that additionally

$$(|\mu| + \varepsilon)^{m+1} < |\mu|^m.$$

Let A_m denote the space of function $g \in A(\Delta_\delta)$ satisfying

$$g(0) = g'(0) = \cdots = g^{(m)}(0) = 0.$$

Estimates similar to those used to established (11) show that if T_m is the restriction of T to A_m, then

$$\|T_m\| \leq (|\mu| + \varepsilon)^{m+1} < |\mu|^m.$$

Now $\lambda I - T_m$ is invertible on A_m for $|\lambda| > \|T_m\|$. Since A_m has codimension $m+1$ in $A(\Delta_\delta)$, T can have at most $m+1$ linearly independent eigenfunctions in $A(\Delta_\delta)$ corresponding to eigenvalues λ satisfying $|\lambda| > \|T_m\|$. Since the functions $f_k(z) = f(z)^k$, $0 \le k \le m$, are linearly independent eigenfunctions with eigenvalues μ^k, $0 \le k \le m$, these must account for all eigenfunctions with eigenvalues $|\lambda| > \|T_m\|$, hence for all eigenfunctions with eigenvalues $|\lambda| \ge |\mu|^m$. $\square$

Solution of Abel's equation. For convenience we denote by $D(C,\beta)$ the sector

$$D(C,\beta) = C + \{|\arg\zeta| < \beta\}$$

with vertex at $C > 0$ and aperture 2β, bisected by the interval $(C,+\infty)$. We are allowed to increase C whenever convenient. We are assuming that $\phi(\zeta)$ maps $D(C,\alpha)$ into itself, and that $\phi(\zeta)$ has the asymptotic form (8). By choosing C sufficiently large, we may assume that

$$\text{Re } \phi(\zeta) > \text{ Re } \zeta + \frac{1}{2}, \qquad \zeta \in D(C,\alpha). \tag{12}$$

We may also assume that for some c_0, $0 < c_0 < \pi/2$, we have $|\arg(\phi(\zeta)-\zeta)| < c_0$, so that the iterates of ζ under $\phi(\zeta)$ remain in the same sector,

$$|\arg(\phi^k(\zeta)-\zeta)| < c_0, \qquad k \ge 1. \tag{13}$$

Fix $\varepsilon > 0$, and consider the subsector $D(C,\alpha-\varepsilon)$. The distance from $\zeta \in D(C,\alpha-\varepsilon)$ to the boundary of $D(C,\alpha)$ is asymptotic to $\varepsilon|\zeta|$ as $\zeta \to \infty$. The Cauchy integral formula for $\phi'(\zeta)$ over a circle centered at $\zeta \in D(C,\alpha-\varepsilon)$ of radius $\varepsilon|\zeta|$ leads to

$$\phi'(\zeta) = 1 + \mathcal{O}(|\zeta|^{-\gamma-1}), \qquad \zeta \in D(C,\alpha-\varepsilon).$$

Consequently for $\zeta \to \infty$ we have the asymptotic estimate

$$\log\phi'(\zeta) = \mathcal{O}(|\zeta|)^{-\gamma-1}, \qquad \zeta \in D(C,\alpha-\varepsilon).$$

If we differentiate Abel's equation (7), we obtain

$$f'(\phi(z))\phi'(z) = f(z),$$

or

$$\log f'(z) - \log f'(\phi(z)) = \log\phi'(z).$$

Thus for the composition operator $(Th)(z) = h(\phi(z))$ as before, and for

$$h(z) \;=\; \log f'(z), \qquad g(z) \;=\; \log\phi'(z),$$

Abel's equation becomes

$$(I-T)h = g.$$

The right-hand side of this resolvent equation satisfies

$$g(z) = \mathcal{O}(|z|^{-\gamma-1}), \qquad \text{as} \quad z \to \infty,$$

and we seek a solution $h(z)$ such that

$$h(z) = \mathcal{O}(|z|^{-\gamma}), \qquad \text{as} \quad z \to \infty.$$

Having such an $h(z)$, we set $f'(z) = e^{h(z)}$ and integrate, thus obtaining a solution of Abel's equation (7) with the asymptotic behavior stated in the theorem.

In this case, the operator $I - T$ is not invertible on any obvious space of analytic functions. However, we may still consider the partial sums of the Neumann series for $(I - T)^{-1}$, and define

$$h_m(z) = (I + T + \cdots + T^m)g.$$

Then

$$(I-T)h_m = g(z) - T^{m+1}g(z) = g(z) - g(\phi^{m+1}(z)).$$

It is now easy to see that $h_m(z)$ converges normally to a solution of $(I-T)h = g$. Indeed, (12) and (13) show that after at most finitely many iterations we are in the right half-plane. Once there, we see from (12) that $|\mathrm{Re}\, \phi^m(z)| \geq m/2$, and

$$g(\phi^m(z)) = \mathcal{O}\left(\frac{1}{|\phi^m(z)|^{\gamma+1}}\right) = \mathcal{O}\left(\frac{1}{m^{\gamma+1}}\right).$$

Since the series $\sum m^{-\gamma-1}$ is summable, $h_m(z)$ converges normally to some analytic function $h(z)$, which satisfies $(I-T)h = g$. To determine the asymptotic behavior of $h(z)$ as $z \to \infty$, we must estimate more carefully. We represent $h(z)$ explicitly as the series

$$h(z) = \sum_{k=0}^{\infty} g(\phi^k(z)),$$

and we break the sum into two pieces. Let S be the sector $\{|\arg \zeta| < \pi/4\}$. The estimate (13) shows that the iterates $\phi^k(\zeta)$ of any ζ eventually enter S. Further, the first integer m_0 for which $\phi^{m_0}(\zeta) \in S$ satisfies $m_0 \leq c_1|\zeta|$ and $|\phi^{m_0}(\zeta)| \geq c_2|\zeta|$, where c_1 and c_2 are independent of ζ. Then

$$(14) \qquad \sum_{k=0}^{m_0-1} |g(\phi^k(\zeta))| \leq c_3 \sum_{k=0}^{m_0-1} |\phi^k(\zeta)|^{-\gamma-1} \leq c_3 m_0 (c_2|\zeta|)^{-\gamma-1} \leq c_4|\zeta|^{-\gamma}.$$

To estimate the terms of the series beyond m_0, note first using (12) that

$$|\phi^n(\xi)| \geq |\xi| + c_5 n, \qquad \xi \in S,$$

and consequently

$$|g(\phi^n(\xi))| \leq \frac{c_6}{|\phi^n(\xi)|^{\gamma+1}} \leq \frac{c_6}{(|\xi| + c_5 n)^{\gamma+1}}, \qquad \xi \in S.$$

With $\xi = \phi^{m_0}(\zeta)$ and $k = m_0 + n$, we then have

$$\sum_{k=m_0}^{\infty} |g(\phi^k(\zeta))| \leq c_6 \sum_{n=0}^{\infty} \frac{1}{(c_2|\zeta| + c_5 n)^{\gamma+1}} \leq c_7 |\zeta|^{-\gamma}. \tag{15}$$

If we add (14) and (15), we obtain $|h(\zeta)| \leq (c_4 + c_7)|\zeta|^{-\gamma}$, as required. □

It is interesting to note that the proof of the existence of a conjugation to a normal form is the easiest in the superattracting case, where the composition operator is quasi-nilpotent, yet historically this was the last of the three cases to be resolved.

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