TENSOR PRODUCTS OF SEMISTABLES ARE SEMISTABLE

BURT TOTARO*

Department of Mathematics, University of Chicago, 5734 S. University Ave., Chicago, IL 60637, USA

Introduction

This paper is devoted to a problem in linear algebra which Faltings formulated and solved using the theory of semistable vector bundles [2]. I will give a somewhat more elementary proof.

Here is the notation we need. Let V be a finite-dimensional vector space over $\mathbb{C}$. A *filtration* of V will mean a decreasing sequence of subspaces V^p of V for $p \in \mathbb{Z}$,

$$\cdots \supset V^{-1} \supset V^0 \supset V^1 \supset \cdots,$$

such that $\cup V^p = V$ and $\cap V^p = \{0\}$. An *N-filtered vector space*, for an integer $N \geq 1$, will mean a vector space V equipped with N filtrations, denoted V_ν^P for $1 \leq \nu \leq N$. Define the *slope* $\mu(V)$ by

$$\mu(V) = \frac{1}{\dim V} \sum_{\nu,p} p \dim \mathrm{gr}_\nu^p(V),$$

where $\mathrm{gr}_\nu^p(V) = V_\nu^p / V_\nu^{p+1}$. For any subspace $W \subset V$, we have the induced filtrations on W, $W_\nu^p = W \cap V_\nu^p$. We say that an N-filtered vector space V is *semistable* if for all $W \subset V$, $W \neq \{0\}$, we have $\mu(W) \leq \mu(V)$. V is *stable* if we have strict inequality $\mu(W) < \mu(V)$ for $W \subset V$, $W \neq \{0\}$, $W \neq V$; and V is *polystable* if it is a direct sum of N-filtered vector spaces, all of the same slope.

For N-filtered vector spaces, we can form $\oplus$, Hom, $\otimes$ as N-filtered vector spaces. For example:

$$(V' \otimes_{\mathbb{C}} V'')_\nu^p = \sum_{p'+p''=p} (V')_\nu^{p'} \otimes (V'')_\nu^{p''}.$$

Theorem 1. *V', V'' semistable $\Rightarrow$ $V' \otimes V''$ semistable. Also, V', V'' polystable $\Rightarrow$ $V' \otimes V''$ polystable.*

Faltings's proof in [2] associates to every N-filtered vector space V a vector bundle E_V over a suitable algebraic curve, with the property that V is semistable $\Leftrightarrow$ E_V is semistable. Then one uses Narasimhan-Seshadri's theorem that a vector bundle on a complex curve is polystable $\Leftrightarrow$ it admits a projectively flat metric. Since the tensor product metric on $E \otimes F$ is projectively flat if E and F are, it follows that the tensor product of polystable bundles is polystable. The same statement for semistable bundles is a corollary. Then at last one returns from vector bundles to N-filtered vector spaces to prove the theorem.

*Supported by NSF grant DMS-9007259. e-mail: totaro@math.uchicago.edu

This paper gives another proof, essentially just using linear algebra over the complex numbers. We find a simple notion of a "good" hermitian metric on an N-filtered vector space V over $\mathbb{C}$ such that V is polystable $\Leftrightarrow$ it admits a good metric. (The proof uses the Kempf-Ness theorem from geometric invariant theory.) It is easy to check that the tensor product of good metrics is good, and the result follows.

Althogh this proof is more algebraic in flavor, it still works only over $\mathbb{C}$. (Of course, semistable N-filtered vector spaces make sense over any field, and those of fixed slope form an abelian category.) Faltings has recently generalized to any field the theorem that the tensor product of semistable N-filtered vector spaces is semistable [1], and the method of [6] gives another proof of this generalization. This is a little surprising, given that the tensor product of semistable bundles over a curve in characteristic p need not be semistable (see for example [5]).

1. Statements

Convension. In what follows we will assume that all N-filtered vector spaces V satisfy $V_\nu^0 = V$, $\nu = 1, \ldots, N$. This can always be arranged by shifting the indexing of the filtrations. In this case, the slope $\mu(V)$ has the simple definition:

$$\mu(V) = \frac{1}{\dim V} \sum_\nu \sum_{p \geq 1} \dim V_\nu^p.$$

Definition. Let V be an N-filtered vector space over $\mathbb{C}$. We say that a hermitian metric on V (that is, on the underlying vector space) is *good* if the sum of all the orthogonal projections $V \to V_\nu^p$ for $\nu = 1, \ldots, N$ and $p \geq 1$ is equal to $\mu(V)$ times the identity endomorphism of V.

Theorem 2. *An N-filtered vector space V over $\mathbb{C}$ is polystable $\Leftrightarrow$ V admits a good metric. Moreover this good metric is unique up to automorphisms of the N-filtered vector space V.*

Theorem 3. *The tensor product of good metrics is good.*

Here Theorem 3 is easy linear algebra, to be proved at the end of the paper. And Theorems 2 and 3 imply the theorem of Faltings.

2. Proof of Theorem 2: Good metric ⇒ polystable

We begin by proving the simpler direction: an N-filtered vector space which admits a good metric is polystable.

So suppose that V has a good metric and let $W \subset V$ be a linear subspace, $W \neq \{0\}$, $W \neq V$. Let i_W denote the inclusion $W \hookrightarrow V$ and $(i_W)^*$ its Hilbert-space adjoint, the orthogonal projection $V \to W$. Similarly let i_ν^p denote the inclusion $V_\nu^p \hookrightarrow V$, for any $1 \leq \nu \leq N$, $p \in \mathbb{Z}$. Consider the composition

$$f_\nu^p = i_W^* i_\nu^p (i_\nu^p)^* i_W.$$

That is, f_ν^p is the endomorphism of $W \subset V$ defined by orthogonally projecting to $V_\nu^p \subset V$ and then orthogonally projecting back to W. Since

$$f_\nu^p = (i_W^* i_\nu^p)(i_W^* i_\nu^p)^*,$$

f_ν^p has the form AA^* and so is a self-adjoint endomorphism of W with nonnegative eigenvalues.

Moreover, f_ν^p is the identity on $W \cap V_\nu^p = W_\nu^p$. So

$$\operatorname{tr} f_\nu^p \geq \dim W_\nu^p,$$

and

$$\begin{aligned} \operatorname{tr} \sum_\nu \sum_{p \geq 1} f_\nu^p &\geq \sum_\nu \sum_{p \geq 1} \dim W_\nu^p \\ &= \mu(W) \dim W. \end{aligned}$$

On the other hand, since the metric on V is good, we know that

$$\sum_\nu \sum_{p \geq 1} i_\nu^p (i_\nu^p)^* = \mu(V) \cdot 1_V.$$

Thus

$$\sum_\nu \sum_{p \geq 1} f_\nu^p = \sum_\nu \sum_{p \geq 1} i_W^* i_\nu^p (i_\nu^p)^* i_W = i_W^* (\mu(V) \cdot 1_V) i_W = \mu(V) \cdot 1_W,$$

and so

$$\operatorname{tr} \sum_\nu \sum_{p \geq 1} f_\nu^p = \mu(V) \dim W.$$

Comparing this with the previous paragraph, we find that $\mu(W) \leq \mu(V)$. That is, we have proved that an N-filtered vector space V with a good metric is at least semistable.

But we can say more. Suppose that V is not stable, so that there is a subspace W with $\mu(W) = \mu(V)$. By the above inequalities, this can only happen if f_ν^p's nonnegative eigenvalues on $(W_\nu^p)^\perp \subset W$ are actually 0, for all p and ν. That is, f_ν^p restricted to $(W_\nu^p)^\perp \subset W$ is 0, which implies (using the definition of f_ν^p as orthogonal projection from W to V_ν^p and back again) that $(W_\nu^p)^\perp \subset W$ is contained in $(V_\nu^p)^\perp \subset V$, for all ν and p. This means that $V = W \oplus W^\perp$ expresses the N-filtered vector space V as a direct sum of N-filtered vector spaces. Moreover, since $\mu(W) = \mu(V)$, we also have $\mu(W^\perp) = \mu(V)$ (because the slope of $W \oplus W^\perp$ is an "average" of the slopes of W and $W^\perp$, namely

$$\mu(W \oplus W^\perp) = \frac{\mu(W) \dim W + \mu(W^\perp) \dim W^\perp}{\dim W + \dim W^\perp}.)$$

And both W and $W^\perp$ have good metrics.

By applying the same argument to W and $W^\perp$, we eventually write any V with a good metric as a direct sum of *stable* N-filtered vector spaces, all of slope $= \mu(V)$. That is, we have proved that if V has a good metric then it is polystable.

3. Proof of Theorem 2: Polystable ⇒ good metric

To show that a direct sum of stable N-filtered vector spaces of the same slope has a good metric, it suffices to show that a stable N-filtered vector space has a good metric.

We begin by explaining the relation of stability for N-filtered vector spaces to Mumford's notion of stable points in a group representation.

Definition. Let G be a reductive algebraic group over $\mathbb{C}$ and W a complex representation of G. A point $x \in W$, $x \neq 0$, is called:

$$\begin{cases} \textit{stable} & \text{if the orbit } Gx \subset W \text{ is closed and } \dim Gx = \dim G, \\ \textit{polystable} & \text{if } Gx \subset W \text{ is closed,} \\ \textit{semistable} & \text{if 0 is not in the closure of } Gx. \end{cases}$$

Let V be an N-filtered vector space. To V we can associate a point in

$$\prod_{\nu=1}^N \prod_{p \geq 1} \mathrm{Gr}_{\dim V_\nu^p}(V),$$

just by $V \mapsto (V_\nu^p : \nu = 1, \ldots, N; p \geq 1)$. (Here $\mathrm{Gr}_k(V)$ is the Grassmannian of k-dimensional subspaces of V.) Moreover, we can embed this product of Grassmannians in the projective space

$$\mathbb{P}(\otimes_{\nu=1}^N \otimes_{p \geq 1} \Lambda^{\dim V_\nu^p} V),$$

so that V determines a line in the vector space

$$W := \otimes_\nu \otimes_{p\geq 1} \Lambda^{\dim V_\nu^p} V.$$

Clearly the group $\mathrm{SL}(V)$ acts on W.

Choose a nonzero point x in the line in W associated to V.

Lemma 1. *An N-filtered vector space V is stable if and only if the point x is stable for the action of $SL(V)$ on W. The same goes for "semistable."*

Proof. We have to show that the orbit $Gx \subset W$ is closed and of maximal dimension $\Leftrightarrow$ for all linear subspaces $U \subset V$ one has

$$\mu(U) < \mu(V),$$

that is,

$$\frac{1}{\dim U}\sum_\nu\sum_{p\geq 1}\dim(U\cap V_\nu^p) < \frac{1}{\dim V}\sum_\nu\sum_{p\geq 1}\dim V_\nu^p.$$

If all of the subspaces $V_\nu^p \subset V$ have the same dimension (or are 0), then this is proved in Mumford-Fogarty [4], p. 88, using the Hilbert-Mumford theorem. The proof in general is the same. This proves the lemma for stable points. Mumford likewise checks that the point x is semistable $\Leftrightarrow$ (in our notation) the N-filtered vector space V is semistable. QED.

Lemma 1 is also true for "polystable" in place of "stable," as will be clear once we prove Theorem 2, but it is inconvenient to prove this now because Mumford and Fogarty do not formulate a polystable version of the Hilbert-Mumford theorem.

Next we give the Kempf-Ness characterization of polystable points in a complex representation.

Theorem 4. (Kempf-Ness [3]). *Let G be a reductive algebraic group over $\mathbb{C}$, W a representation of G, $x \in W$ a nonzero vector. Choose a maximal compact subgroup $H \subset G$ and an H-invariant hermitian metric on W. Then the following are equivalent:*

(1) *x is polystable: that is, the orbit $Gx \subset W$ is closed.*
(2) *The function $d_x\colon G \to \mathbb{R}$ defined by*

$$d_x(g) = |gx|^2$$

(using the metric on W) attains its minimum on G.
(3) *The function d_x has a critical point on G.*

More precisely, any critical point of d_x gives actually a global minimum, and the set of critical points (if nonempty) is just one double coset in $H\backslash G/G_x$, where G_x is the isotropy group of x.

Of course (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is trivial. The converse is an elegant application of the Cartan decomposition of G, which in turn is just classical linear algebra in the case $G = \mathrm{SL}(V)$.

For $G = \mathrm{SL}(V)$, one can restate the Kempf-Ness theorem as follows, using the fact that $\mathrm{SL}(V)$ acts transitively on the space $M_\omega(V)$ of hermitian metrics on V with a given volume form ω.

Corollary 1. *Let V be a vector space over $\mathbb{C}$ with a volume form ω. Let $W \subset \otimes^r V$ be a representation of $\mathrm{SL}(V)$, $x \in W$ a nonzero vector. Then the following are equivalent:*

(1) *x is polystable, that is, the orbit $\mathrm{SL}(V)\cdot x \subset W$ is closed.*
(2) *There is a hermitian metric m_0 on V with volume form ω which minimizes the function*

$$d_x(m) = |x|_m^2$$

on M_ω. (Here $|x|_m$ denotes the length of x in the metric on $W \subset \otimes^r V$ associated to a metric on V.)
(3) *There is a hermitian metric m_0 on V which is a critical point for the function $d_x : M_\omega(V) \to \mathbb{R}$.*

Moreover, if there is a metric on V as in (3), it is unique up to the action of the subgroup of $\mathrm{SL}(V)$ which fixes the point $x \in W$.

We want to use this criterion when V is an N-filtered vector space and $x \in \otimes_{\nu=1}^N \otimes_{p\geq 1} \Lambda^{\dim V_\nu^p} V$ is a point in the line associated to V. That is, if we choose a basis $\alpha_{\nu,i}^p \in V$, $1 \leq i \leq \dim V_\nu^p$, for each subspace V_ν^p, then we can take

$$x = (\alpha_{1,1}^1 \wedge \cdots \wedge \alpha_{1,\dim V_1^1}^1) \otimes (\alpha_{2,1}^1 \wedge \cdots \wedge \alpha_{2,\dim V_2^1}^1) \otimes \cdots .$$

It follows that, for any metric m on V,

$$|x|_m = \prod_\nu\prod_{p\geq 1} \mathrm{vol}_m(\alpha_{\nu,1}^p \wedge \cdots \alpha_{\nu,\dim V_\nu^p}^p).$$

More simply, if we have a pair of metrics m_0 and m on V, we can say that

$$\frac{|x|_m}{|x|_{m_0}} = \prod_\nu\prod_{p\geq 1}\left(\frac{\mathrm{vol}_m(V_\nu^p)}{\mathrm{vol}_{m_0}(V_\nu^p)}\right).$$

Here both metrics give volume forms on all linear subspaces V_ν^p, and the ratio of the volume forms on each V_ν^p is a positive real number.

By Lemma 1 and Corollary 1, every stable N-filtered vector space V has a metric m_0, with associated volume form ω, such that m_0 is a critical point for the function

$$\begin{aligned} d_x(m) =& |x|_m^2 \\ =& \prod_\nu \prod_{p\geq 1} \mathrm{vol}_m(\alpha^p_{\nu,1} \wedge \cdots \alpha^p_{\nu,\dim V^p_\nu}) \end{aligned}$$

on $M_\omega(V)$. We can assume here that $\alpha^p_{\nu,1}, \ldots, \alpha^p_{\nu,\dim V^p_\nu}$ is an orthonormal basis for V^p_ν with respect to the metric m_0. The tangent space to $M_\omega(V)$ at m_0 can be identified with the space of self-adjoint endomorphisms b of (V, m_0) with trace 0.

In these terms,

$$d(\mathrm{vol}_m(\alpha^p_{\nu,1} \wedge \cdots \wedge \alpha^p_{\mu,\dim V^p_\nu})^2)|_{m_0}(b) \;=\; d(\det < \alpha^p_{\nu,i}, \alpha^p_{\nu,j} >_m)|_{m_0}(b).$$

Using the fact that $< \alpha^p_{\nu,i}, \alpha^p_{\nu,j} >_{m_0} = \delta_{ij}$, we can evaluate the derivative of the determinant as a trace:

$$\begin{aligned} &= \sum_{i=1}^{\dim V^p_\nu} d(< \alpha^p_{\nu,i}, \alpha^p_{\nu,i} >_m)|_{m_0}(b) \\ &= \mathrm{tr}(\pi^p_\nu \circ b), \end{aligned}$$

where π^p_ν denotes the orthogonal projection with respect to the metric m_0 from V onto V^p_ν. We view π^p_ν as an endomorphism of V. (In the previous section's notation, we could write $\pi^p_\nu = i^p_\nu (i^p_\nu)^*$.)

Since all the volumes $\mathrm{vol}_m(\alpha^p_{\nu,1} \wedge \cdots)$ are equal to 1 when $m = m_0$, the derivative at m_0 of the product of these volumes is just the sum of the derivatives. That is:

$$\begin{aligned} d(d_x)|_{m_0}(b) \;&=\; d(\prod_\nu \prod_{p\geq 1} \mathrm{vol}_m(\alpha^p_{\nu,1} \wedge \cdots \wedge \alpha^p_{\nu,\dim V^p_\nu})^2)|_{m_0}(b) \\ &= \sum_\nu \sum_{p\geq 1} \mathrm{tr}(\pi^p_\nu \circ b) \\ &= \mathrm{tr}(\sum_\nu \sum_{p\geq 1} \pi^p_\nu \circ b). \end{aligned}$$

So the metric m_0 is a critical point for the function $d_x : M_\omega(V) \to \mathbb{R} \iff$

$$\mathrm{tr}(\sum_\nu \sum_{p\geq 1} \pi^p_\nu \circ b) = 0$$

for all self-adjoint endomorphisms b of (V, m_0) with trace 0. Since $\mathrm{tr}(AB)$ is a nondegenerate bilinear form on the space of self-adjoint endomorphisms, and $\sum_\nu \sum_{p\geq 1} \pi^p_\nu$ is self-adjoint, this condition is equivalent to

$$\sum_\nu \sum_{p\geq 1} \pi^p_\nu \in \mathbb{R} \cdot 1_V.$$

By computing the trace of this endomorphism (namely, $\sum_\nu \sum_{p\geq 1} \dim V^p_\nu$), one sees at last that m_0 is a critical point for the function d_x if and only if

$$\begin{aligned} \sum_\nu \sum_{p\geq 1} \pi^p_\nu &= (\frac{1}{\dim V} \sum_\nu \sum_{p\geq 1} \dim V^p_\nu) \cdot 1_V \\ &= \mu(V) \cdot 1_V. \end{aligned}$$

That is, m_0 is a critical point for $d_x : M_\omega(V) \to \mathbb{R} \iff m_0$ is a good metric in the sense defined before Theorem 2. It follows that every stable N-filtered vector space has a good metric. We deduce that every polystable N-filtered vector space, being a direct sum of stable spaces of the same slope, has a good metric.

The uniqueness part of the Kempf-Ness theorem implies that a good metric, when it exists, is unique up to automorphisms of the N-filtered vector space V.

QED, Theorem 2.

4. Proof of Theorem 3

Let V, W be N-filtered vector spaces with good metrics. We want to show that the induced metric on the N-filtered vector space $V \otimes W$ is also good. Let π^p_ν and ρ^p_ν be the orthogonal projections $V \to V^p_\nu$ and $W \to W^p_\nu$, respectively. For each ν and p, the orthogonal projection

$$\sigma^p_\nu \colon V \otimes W \to (V \otimes W)^p_\nu = \sum_{p'+p''=p} V^{p'}_\nu \otimes W^{p''}_\nu$$

is given by

$$\sigma^p_\nu = \sum_{i\in\mathbb{Z}} \pi^i_\nu \otimes \rho^{p-i}_\nu - \sum_{i\in\mathbb{Z}} \pi^i_\nu \otimes \rho^{p+1-i}_\nu.$$

(For example, one can check this for each ν by choosing orthonormal bases for V and W which are compatible with the filtrations $V^\cdot_\nu$ and $W^\cdot_\nu$.) We have $\pi^i_\nu = 0$ and $\rho^i_\nu = 0$ for i sufficiently large. It follows that

$$\begin{aligned} \sum_\nu \sum_{p\geq 1} \sigma^p_\nu &= \sum_\nu \sum_{i\in\mathbb{Z}} \pi^i_\nu \otimes \rho^{1-i}_\nu = \sum_\nu \sum_{i\geq 1} \pi^i_\nu \otimes \rho^{1-i}_\nu + \sum_\nu \sum_{i\leq 0} \pi^i_\nu \otimes \rho^{1-i}_\nu \\ &= \sum_\nu \sum_{i\geq 1} \pi^i_\nu \otimes 1_W + \sum_\nu \sum_{i\leq 0} 1_V \otimes \rho^{1-i}_\nu \\ &= \mu(V) \cdot 1_V \otimes 1_W + \mu(W) \cdot 1_V \otimes 1_W = \mu(V \otimes W) \cdot 1_{V\otimes W}. \end{aligned}$$

That is, the tensor product of good metrics is good. QED, Theorem 3.

References

1. G. Faltings., *Mumford-Stabilität in der algebraischen Geometrie*, To appear in the 1994 ICM proceedings.
2. G. Faltings and G. Wüstholz, *Diophantine approximations in projective spaces*, Invent. Math. **116** (1994), 109–138.
3. G. Kempf and L. Ness, *The length of vectors in representation spaces*, in Algebraic Geometry, Copenhagen 1978, Springer Lecture Notes in Math. **732** (1979).
4. D. Mumford and J. Forgaty, *Geometric Invariant Theory* (Springer, Heidelberg 1965).
5. S. Ramanan and A. Ramanathan, *Some remarks on the instability flag*, Tohoku Math. J. **36** (1984), 269–291.
6. B. Totaro, *Tensor products of weakly admissible filtered isocrystals* (*as used in p-adic Hodge theory*), preprint.