Tensor products in p-adic Hodge theory

Burt Totaro *

There is a classical relation between the p-adic absolute value of the eigenvalues of Frobenius on crystalline cohomology and Hodge numbers, for a variety in characteristic p: "the Newton polygon lies on or above the Hodge polygon" [14], [1]. For a variety in characteristic p with a lift to characteristic 0, Fontaine conjectured and Faltings proved a more precise statement: there is an inequality which relates the slope of Frobenius on any Frobenius-invariant subspace of the crystalline cohomology to the Hodge filtration, restricted to that subspace [7], [4]. A vector space over a p-adic field together with a σ-linear endomorphism and a filtration which satisfies this inequality is called a weakly admissible filtered isocrystal (see section 1 for the precise definition). The category of such objects is one possible p-adic analogue of the category of Hodge structures: in particular, it is an abelian category.

We give a new proof of Faltings's theorem that the tensor product of weakly admissible filtered isocrystals over a p-adic field is weakly admissible [5]. By a similar argument, we also prove a characterization of weakly admissible filtered isocrystals with G-structure in terms of geometric invariant theory, which was conjectured by Rapoport and Zink [19]. Before Faltings, Laffaille [12] had proved the tensor product theorem in the case of filtered isocrystals over an unramified extension of $\mathbf{Q}_p$.

Faltings's proof works by reducing this problem of σ-linear algebra to a different problem of pure linear algebra, the problem of showing that the tensor product of two vector spaces, each equipped with a finite "semistable" set of filtrations, is semistable. The latter problem is solved by constructing suitable integral lattices (in [5]) or hermitian metrics (in [20]) on vector spaces with a semistable set of filtrations, just as one can prove that the tensor product of semistable bundles on an algebraic curve is semistable using Narasimhan-Seshadri's hermitian metrics ([6], [16]). In this paper, we can avoid the reduction from filtered isocrystals to filtered vector spaces.

The point is that Ramanan and Ramanathan's algebraic proof [17] that the tensor product of semistable vector bundles is semistable can be modified to apply directly to filtered isocrystals. We have an inequality to prove for a class of linear subspaces S of a tensor product $V \otimes W$. The inequality is obvious for sufficiently general subspaces S and also if S is a very special subspace, say if S is a decomposable subspace $S_1 \otimes S_2 \subset V \otimes W$. But it is not clear how to prove the inequality we want if S is somewhere in the middle. The solution, following Ramanan and Ramanathan, is to use geometric invariant theory to give a sharp dichotomy between "general" subspaces and "special" subspaces of $V \otimes W$, in such a way that we get useful information in either case.

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Section 1 defines filtered isocrystals and explains how they arise geometrically. Sections 2 and 3 explain the ideas from geometric invariant theory which are used in the proof. Section 4 proves the tensor product theorem, and section 5 generalizes it to some bigger categories of filtered objects (involving a nilpotent "logarithm of the monodromy" endomorphism) which Fontaine defined. Finally, sections 6-8 prove the characterization of weakly admissible filtered isocrystals with G-structure that Rapoport and Zink conjectured.

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1 Background of the problem

We review Fontaine's theory, which relates Galois representations to filtered isocrystals [7].

Let K be a p-adic field (a finite extension of $\mathbf{Q}_p$). Fontaine defined an abelian $\otimes$-category of representations of the Galois group $Gal(\overline{K}/K)$ over $\mathbf{Q}_p$ called the category of crystalline Galois representations, as well as a $\otimes$-functor from the category of crystalline Galois representations of $Gal(\overline{K}/K)$ over $\mathbf{Q}_p$ to the additive $\otimes$-category of filtered isocrystals over K, as defined below. The functor is exact and fully faithful. The interest of this functor is that, if X is a smooth projective variety over K with good reduction, so that X extends to a smooth proper scheme X/o_K, the functor takes the Galois representation on the p-adic etale cohomology of X, $H^i(X \otimes_K \overline{K}, \mathbf{Q}_p)$, to the crystalline cohomology $H^i_{\text{crys}}(X/W(k)) \otimes K_0$ together with the de Rham filtration on $H^i_{\text{DR}}(X/K) = H^i_{\text{crys}}(X/W(k)) \otimes K$ [4]. Thus the full faithfulness of Fontaine's functor implies that p-adic etale cohomology gives exactly the same information about X as crystalline plus de Rham cohomology. The latter information, the filtered isocrystal, describes in some sense the "variation of Hodge structures" associated to the one-parameter family of schemes X/o_K.

Define a filtered isocrystal to be admissible if it is isomorphic to the image of a crystalline Galois representation under the functor. Thus the functor gives an equivalence of abelian $\otimes$-categories between crystalline Galois representations and admissible filtered isocrystals. Fontaine defined a rather simple abelian subcategory of the additive category of filtered isocrystals called weakly admissible filtered isocrystals (defined below), such that every admissible filtered isocrystal is weakly admissible, and he conjectured that weakly admissible implies admissible. He and Laffaille proved this when K is unramified over $\mathbf{Q}_p$ and the filtration on the isocrystal has length $\leq p-2$ [9].

Fontaine's conjecture would imply that the tensor product of weakly admissible filtered isocrystals is weakly admissible. This consequence of Fontaine's conjecture is now known to be true, as Faltings proved and as we will prove in this paper. One consequence of the theorem is that one can define the structure group of a weakly admissible filtered isocrystal, using the theory of tannakian categories. The space of weakly admissible filtered isocrystals with a given structure group is a rigid analytic space analogous to Griffiths's period domains for Hodge structures [19].

We now define the additive $\otimes$-category of filtered isocrystals over K. Here K

is a complete, discretely valued extension field of $\mathbf{Q}_p$ with perfect residue field k. This is more general than just a finite extension of $\mathbf{Q}_p$. Let K_0 be the quotient field of the Witt ring $W(k)$ of the residue field k; $K_0 \subset K$ is the largest unramified extension of $\mathbf{Q}_p$ contained in K, and K is a finite extension of K_0. Let σ denote the canonical lift of the Frobenius automorphism of k to an automorphism of K_0. An isocrystal over K_0 is a finite-dimensional vector space V_0 over K_0 together with a bijective σ-linear endomorphism φ of V_0 ($\varphi(x+y) = \varphi x + \varphi y$, $\varphi(ax) = a^\sigma \varphi x$ for $a \in K_0$). The isocrystals over K_0 form a $\mathbf{Q}_p$-linear abelian category. An isocrystal has a natural grading (by "slopes")

$$V_0 = \oplus_{l \in \mathbf{Q}} (V_0)_l,$$

which we define in Lemma 1 below. We write $\text{ord}_p(\det \varphi)$ for $\sum l \dim (V_0)_l$.

A filtered isocrystal V over K is an isocrystal V_0 over K_0 together with a filtration of the K-vector space $V := V_0 \otimes_{K_0} K$ indexed by the integers, V^i for $i \in \mathbf{Z}$, such that $V^i \supset V^j$ for $i \leq j$, $V^i = V$ for $i << 0$, and $V^i = 0$ for $i >> 0$. Later it will be convenient to allow also filtrations of V indexed by the rational numbers; see section 2. (The name "filtered isocrystal" is not very good, since the filtration (V^i) is just a filtration by K-linear spaces, not by sub-isocrystals.)

Now we can define weakly admissible filtered isocrystals.

Definition. A filtered isocrystal V is weakly admissible $\iff$ for every φ-invariant subspace $W_0 \subset V_0$, if we let $W = W_0 \otimes_{K_0} K \subset V$, and $W^q = W \cap V^q$, then

$$\sum_q q \dim (W^q/W^{q+1}) \leq \text{ord}_p(\det \varphi | W_0),$$

with equality for $W_0 = V_0$.

To conclude this section of the paper, here is the lemma needed to define the slope grading of an isocrystal.

Lemma 1 *Let k be a perfect field of characteristic $p > 0$, and let K_0 be the fraction field of the ring of Witt vectors of k. For any isocrystal V_0 over K_0, let $(V_0)_l$, for a rational number $l = r/s$, $(r,s) = 1$, $s > 0$, be the largest linear subspace of V_0 which contains a $W(k)$-lattice M with*

$$\varphi^s M = p^r M.$$

Then $(V_0)_l$ is well-defined and $V = \oplus (V_0)_l$.

Here, by definition, a lattice in a K_0-vector space V_0 is a finitely generated $W(k)$-submodule of V_0 which spans V_0 as a vector space.

Proof. Since the sum of two subspaces of V_0 which both contain lattices with $\varphi^s M = p^r M$ is another subspace with such a lattice (namely the sum of the two lattices), it is clear that there is a unique largest subspace $(V_0)_{r/s}$ containing such a lattice. Thus $(V_0)_{r/s}$ is well-defined.

When the residue field k is algebraically closed, Dieudonné classified the isocrystals over K_0 up to isomorphism: they are direct sums of the irreducible isocrystals over K_0, which are indexed by the rational numbers: for $r/s \in \mathbf{Q}$, $(r,s) = 1$, $s > 0$, the corresponding irreducible isocrystal is $A_{r/s} = (K^0)^s$, with

$$\varphi(x_1, \cdots, x_s) = (p^r \sigma(x_s), \sigma(x_1), \cdots, \sigma(x_{s-1})).$$

One proof is given in [13]. Moreover, the splitting of an isocrystal as a sum $V = \oplus_{l\in\mathbf{Q}}(V_0)_l$, where here $(V_0)_l$ denotes a direct sum of copies of A_l, is unique. If k is merely perfect, an isocrystal over $K_0(k)$ can be tensored up to give an isocrystal over $K_0(\overline{k})$, which is invariant by the obvious action of $Gal(\overline{k}/k)$. The isocrystal over $K_0(\overline{k})$ has a splitting as above, which is $Gal(\overline{k}/k)$-invariant, and therefore comes from a splitting $V_0 = \oplus_{l\in\mathbf{Q}}(V_0)_l$ of the original isocrystal over $K_0(k)$. One checks easily, from the explicit description of the isocrystals A_l for k algebraically closed, that the subspace $(V_0)_l$ so defined is the largest subspace of V_0 which contains a lattice M with $\varphi^s M = p^r M$. QED.

2 The ideas from geometric invariant theory

Definition. Let V be a finite-dimensional vector space over a field K. A filtration α of V is a decreasing set of subspaces labeled by the rational numbers, V^i_α for $i \in \mathbf{Q}$, such that $V^i = V$ for $i << 0$, $V^i \supset V^j$ for $i \leq j$, $V^i = \cap_{j\in\mathbf{Q},j<i}V^j$, and $V^i = 0$ for $i >> 0$. We let $\mathrm{gr}^i_\alpha V = V^i/V^{i+\epsilon}$ for sufficiently small $\epsilon > 0$; this actually makes sense for any $i \in \mathbf{R}$. We assume that $\mathrm{gr}^i_\alpha V$ is nonzero only for rational numbers i.

The degree $\deg_\alpha(V)$ is defined to be $\sum_{i\in\mathbf{Q}} i \dim \mathrm{gr}^i V$, and the slope $\mu_\alpha(V)$ is $\deg_\alpha(V)/\dim(V)$. A subquotient of a filtered vector space inherits a natural filtration: if $S \subset V$, then we define $S^i = S \cap V^i$, and if V maps onto Q, then we set $Q^i = \mathrm{im}\, V^i$. If V and W are filtered vector spaces, then we give $V \otimes W$ the filtration

$$(V\otimes W)^i = \sum_j V^j \otimes W^{i-j}.$$

Finally, I sometimes say "a filtration α of (V,W)" to mean "a filtration (V^i_α) of V and a filtration (W^i_α) of W."

Definition. Let V and W be vector spaces over a field K. We say that a subspace $S \subset V \otimes W$ is $GL(V) \times GL(W)$-semistable if for every filtration α of (V,W), we have $\mu(S) \leq \mu(V \otimes W)$. ($S$ gets a filtration as a subspace of $V \otimes W$.)

If the field K happens to be perfect, then this definition is equivalent to semistability in the sense of geometric invariant theory [15], as explained in the following lemma. We will not use this lemma elsewhere in the paper, however; we just need the elementary definition above, and there is no need to restrict to perfect fields K in the rest of this section.

Lemma 2 *Suppose that K is a perfect field. Let V, W be K-vector spaces of dimensions v, w. Then the above definition of $GL(V)\times GL(W)$-semistability of an s-dimensional subspace $S \subset V \otimes W$ is equivalent to the semistability of the point $[S]$ in the Grassmannian $Gr_s(V \otimes W)$ with respect to the action of $GL(V) \times GL(W)$ and the $GL(V) \times GL(W)$-line bundle $S \mapsto ((\Lambda^s S)^*)^{\otimes vw} \otimes (\Lambda^{vw}(V \otimes W))^{\otimes s}$.*

The line bundle mentioned is merely the simplest $GL(V) \times GL(W)$-equivariant ample line bundle on the Grassmannian on which the center of $GL(V) \times GL(W)$ acts trivially.

Proof. By the Hilbert-Mumford theorem, generalized by Kempf from algebraically closed fields to all perfect fields, semistability of a point $[S]$ in the Grassmannian with respect to the above $GL(V) \times GL(W)$-line bundle L is equivalent to the nonnegativity of a certain number $\mu(S, \lambda, L)$ for all one-parameter subgroups $\alpha : G_m \to GL(V) \times GL(W)$ defined over K ([15], p. 49). We have

$$\mu(S, \lambda, L) = vw\ \mu(S, \lambda, (\Lambda^s S)^*) + s\ \mu(S, \lambda, \Lambda^{vw}(V \otimes W))$$

because $\mu(S, \lambda, L)$ is a group homomorphism $\text{Pic}^G(X) \to \mathbf{Z}$ as a function of L. Since the action of $GL(V) \times GL(W)$ on the Grassmannian and on these line bundles extends to $GL(V \otimes W)$, the numbers $\mu(S, \lambda, (\Lambda^s S)^*)$ and $\mu(S, \lambda, \Lambda^{vw}(V \otimes W))$ are actually defined for all one-parameter subgroups λ of $GL(V \otimes W)$. The number $\mu(S, \lambda, (\Lambda^s S)^*)$ is computed in [15], pp. 87-88, for any vector space in place of $V \otimes W$:

$$\begin{aligned} \mu(S, \lambda, (\Lambda^s S)^*) &= -\text{deg}_\lambda S \\ &= -s\ \mu_\lambda S \end{aligned}$$

(equivalent to the second-to-last displayed equation on p. 87). Here $\mu_\lambda S$ refers to the decreasing filtration of $V \otimes W$ provided by the one-parameter subgroup λ and the resulting subspace filtration on S. Actually Mumford only states this for one-parameter subgroups of SL, but it holds with the same proof for one-parameter subgroups of GL.

Since $\Lambda^{vw}(V \otimes W)$ is just a trivial line bundle on the Grassmannian with $GL(V \otimes W)$ acting by a character, it is easy to compute from the definition that

$$\begin{aligned} \mu(S, \lambda, \Lambda^{vw}(V \otimes W)) &= \text{deg}_\lambda(V \otimes W) \\ &= vw\ \mu_\lambda(V \otimes W). \end{aligned}$$

Thus the point $[S] \in Gr_s(V \otimes W)$ is semistable with respect to the action of $GL(V) \times GL(W)$ and the line bundle $L = ((\Lambda^s S)^*)^{\otimes vw} \otimes (\Lambda^{vw}(V \otimes W))^{\otimes s}$

$\iff$ $\mu(S, \lambda, L) \geq 0$ for all one-parameter subgroups λ of $GL(V) \times GL(W)$

$\iff$ $vw\ \mu(S, \lambda, (\Lambda^s S)^*) + s\ \mu(S, \lambda, \Lambda^{vw}(V \otimes W)) \geq 0$ for all one-parameter subgroups λ of $GL(V) \times GL(W)$

$\iff$ $svw(\mu_\lambda S - \mu_\lambda(V \otimes W)) \leq 0$ for all filtrations λ of V and W (since this condition only depends on the filtration of $V \otimes W$ associated to the one-parameter subgroup λ, thus only on the filtrations of V and W associated to λ)

$\iff$ S is semistable in the sense of the above definition. QED.

The following two crucial propositions will be proved in the next section. These are just special cases of some general results on geometric invariant theory due to Kempf [11] and Ramanan and Ramanathan ([17], Prop. 1.12), but I want to show how elementary the proofs are in the case that we need.

Proposition 1 *Let V, W be vector spaces over a field K. Let $S \subset V \otimes W$ be a subspace which is not $GL(V) \times GL(W)$-semistable. Then there is a filtration of (V, W) which maximizes the function*

$$f(S, \alpha) = \frac{\mu S - \mu(V \otimes W)}{|\alpha|}$$

on the set of nontrivial filtrations $\alpha = (V^i, W^i)$ of (V, W) indexed by $i \in \mathbf{Q}$. Here we define

$$|\alpha| = (\sum i^2 dim\ gr^i V + \sum i^2 dim\ gr^i W)^{1/2},$$

and a nontrivial filtration α of (V,W) just means a filtration which is not $V^i = V$, $W^i = W$ for $i \leq 0$ and $V^i = 0$, $W^i = 0$ for $i > 0$; this ensures that the denominator $|\alpha|$ is positive.

Moreover, the maximizing filtration (called Kempf's filtration $\alpha(S)$ associated to S) is unique up to scaling (new V^i equals old V^{ki} and new W^i equals old W^{ki}, $k \in \mathbf{Q}$, $k > 0$). Finally, Kempf's filtration has $\mu V = \mu W = 0$.

Proposition 2 *Let V, W be vector spaces over a field K, and let $S \subset V \otimes W$ be a subspace which is not $GL(V) \times GL(W)$-semistable. Let $\alpha(S) = (V^i_{\alpha(S)}, W^i_{\alpha(S)})$ be Kempf's filtration of (V,W) associated to S (it is unique up to scaling, so just pick one). Then there is a constant $c > 0$ such that for every filtration β of (V,W), one has*

$$\mu_\beta S - \mu_\beta(V \otimes W) \leq c \left(\int (\mu_\beta V^l_{\alpha(S)} - \mu_\beta V) \ dim\ V^l_{\alpha(S)} dl + \int (\mu_\beta W^l_{\alpha(S)} - \mu_\beta W) \ dim\ W^l_{\alpha(S)} dl \right).$$

The integrals are integrals of piecewise constant functions on the real line which are 0 at infinity.

3 Proof of Propositions 1 and 2

We just write out Ramanan and Ramanathan's proofs, which are written in the language of geometric invariant theory over a general reductive group, for the case at hand. We omit the proof of the following easy lemma of convex geometry, however [17].

Lemma 3 *Let T be a finite nonempty set of linear forms on $\mathbf{R}^n$. Define, for $x \in \mathbf{R}^n$, $f(x) = \inf\{l(x) : l \in T\}$. Then $f(cx) = cf(x)$ for $c > 0$ and f is a concave function. Let $S^{n-1} \subset \mathbf{R}^n$ be the unit sphere. Suppose that $f(x) > 0$ for some $x \in \mathbf{R}^n$.*

Then f restricted to S^{n-1} attains its maximum at a unique point a on the sphere. If the linear forms in T have rational coefficients, then $c \cdot a \in \mathbf{Z}^n$ for some real $c > 0$. If $f = 0$ on some linear subspace $A \subset \mathbf{R}^n$, then the maximum point a is orthogonal to A.

Also, let $\langle\ ,\ \rangle$ be the usual inner product on $\mathbf{R}^n$. Let $c = f(a) > 0$. Then

$$f(x) \leq c\langle a, x\rangle$$

for all $x \in \mathbf{R}^n$.

The first thing we will actually prove is the following lemma. Consider a decomposition of a vector space V as a direct sum of lines, $V = L_1 \oplus \cdots \oplus L_v$, $v = \dim V$. We say that a filtration (V^i) of V is compatible with such a splitting of V if there are rational numbers $(i_1, \ldots, i_n)$ such that V^i is the sum of the lines L_j such that $i_j \geq i$. The set of filtrations compatible with a given splitting of V is naturally in one-to-one correspondence with $\mathbf{Q}^v$ via the numbers $(i_1, \ldots, i_n)$. The following lemma explains how the degree of a given subspace $S \subset V$ (with the filtration induced from V) depends on $i_1, \ldots, i_n$.

Lemma 4 *Let V be a vector space which is decomposed as a direct sum of lines, $V = L_1 \oplus \cdots \oplus L_v$. For a subspace $S \subset V$ of dimension s, let $\mathbf{S}$ be the set of subsets $A \subset \{1, \ldots, v\}$ of order s such that the coordinate projection $S \to V_A$ is an isomorphism. Then, with respect to any filtration of V compatible with the given splitting,*

$$deg\ S = \inf_{A \in \mathbf{S}} deg\ V_A.$$

In other words, for the filtration given by rational numbers $i_1, \ldots, i_v$,

$$deg\ S = \inf_{A \in \mathbf{S}} \sum_{j \in A} i_j.$$

Proof. For any subset $A \subset \{1, \ldots, v\}$, the projection $S \to V_A$ is compatible with filtrations, so if it is also an isomorphism of vector spaces then $\deg S \leq \deg V_A$. We have to find at least one V_A such that $S \to V_A$ is an isomorphism of filtered vector spaces, so that $\deg S = \deg V_A$. For each $i \in \mathbf{Q}$, $\mathrm{gr}^i S$ is a subspace of $\mathrm{gr}^i V$, which is a vector space with a splitting given by the lines L_j such that $i_j = i$. So for each i we can choose a subset A_i of the set of j's with $i_j = i$, such that $\mathrm{gr}^i S$ projects isomorphically to S_{A_i}. If we let $A \subset \{1, \ldots, v\}$ be the union of all the A_i, then the projection $S \to V_A$ is compatible with filtrations, and $\mathrm{gr}^* S$ maps isomorphically to $\mathrm{gr}^* V_A = \oplus_i S_{A_i}$, so that $S \to V_A$ is an isomorphism of filtered vector spaces. QED.

Proof of Proposition 1. Let V and W be vector spaces over a field K, and let $S \subset V \otimes W$ be a subspace which is not $GL(V) \times GL(W)$-semistable. This means that there is a filtration α of (V, W) such that $f(S, \alpha) > 0$.

Fix splittings of the vector spaces V and W. The set of filtrations of (V, W) compatible with the given splitting of (V, W) is naturally in one-to-one correspondence with $\mathbf{Q}^{v+w}$. Put the usual inner product on $\mathbf{Q}^{v+w}$. The function $f(S, \alpha)$ restricted to this set of filtrations α is the infimum of a finite set of linear forms with rational coefficients, divided by the norm of α. Indeed, let $s = \dim S$, and let $\mathbf{S}$ be the set of all s-element subsets $A = \{(a_1, b_1), \ldots, (a_s, b_s)\}$ of $\{1, \ldots, v\} \times \{1, \ldots, w\}$ such that the projection map from S to the coordinate subspace $(V \otimes W)_A \subset V \otimes W$ is surjective. Then, for a filtration α described by rational numbers $(i_1, \ldots, i_v)$ and $(j_1, \ldots, j_w)$, Lemma 4 shows that the degree of S with respect to the resulting filtration of $V \otimes W$ is

$$\deg S = \inf_{A \in \mathbf{S}} (i_{a_1} + j_{b_1}) + \cdots + (i_{a_s} + j_{b_s}).$$

By Lemma 3, if there is a filtration α compatible with the given splitting such that $f(S, \alpha) > 0$, then there is a filtration α which maximizes $f(S, \alpha)$ among the filtrations compatible with the given splitting, and it is unique up to scaling.

Now we go beyond a fixed splitting. We have assumed that there is a filtration α of (V, W) with $f(S, \alpha) > 0$. The function $f(S, \alpha)$ on the set of filtrations α compatible with a given splitting of (V, W) only depends on which subsets $A \subset \{1, \ldots, v\} \times \{1, \ldots, w\}$ have the projection $S \to (V \otimes W)_A$ an isomorphism. So there are only finitely many possible functions on $\mathbf{Q}^{v+w}$ that arise this way from splittings of (V, W). It follows that there is a filtration α which maximizes $f(S, \alpha)$ among all nontrivial filtrations of (V, W).

Suppose that α and β are filtrations of (V,W) which achieve this maximum value of $f(S,\cdot)$. It is an easy fact of linear algebra (the Bruhat decomposition) that there is a splitting of (V,W) which is compatible with both α and β. By our earlier uniqueness statement, we deduce that α and β are the same up to scaling.

This proves the main part of Proposition 1. To see that the maximizing filtration of (V,W) satisfies $\mu V = \mu W = 0$, notice that $f(S,\alpha) = 0$ for all "constant" filtrations α of (V,W) (meaning that $V^i = V$, $V^{i+\epsilon} = 0$, $W^j = W$, $W^{j+\epsilon} = 0$ for some $i, j \in \mathbf{Q}$), and apply the second paragraph of Lemma 3. QED.

Proof of Proposition 2. Let V and W be vector spaces over a field K and let $S \subset V \otimes W$ be a subspace which is not $GL(V) \times GL(W)$-semistable (as defined above). Let $\alpha = \alpha(S)$ be Kempf's filtration of (V,W) associated to S (it is unique up to scaling, so just pick one). Let β be an arbitrary filtration of (V,W). Then we can choose a splitting of (V,W) which is compatible with both β and Kempf's filtration α; that is, both filtrations are described by points of $\mathbf{Q}^{v+w}$ in terms of the chosen splitting, and Kempf's filtration maximizes the function

$$f(S,\gamma) = \frac{\mu_\gamma S - \mu_\gamma(V \otimes W)}{|\gamma|}$$

on the set of $\gamma \in \mathbf{Q}^{v+w} - 0$ (since it maximizes f among all filtrations). By Lemma 3, there is a constant $c = (\mu_\alpha S - \mu_\alpha(V \otimes W))/|\alpha|^2 > 0$, clearly independent of β and the choice of splitting made above, such that

$$\begin{aligned}
\mu_\beta S - \mu_\beta(V \otimes W) &\leq c\langle \beta, \alpha\rangle \\
&= c\left(\sum_{k,l \in \mathbf{Q}} kl \dim \mathrm{gr}^k_\beta \mathrm{gr}^l_\alpha V + \sum_{k,l \in \mathbf{Q}} kl \dim \mathrm{gr}^k_\beta \mathrm{gr}^l_\alpha W\right) \\
&= c\left(\sum_{l \in \mathbf{Q}} l\ \mu_\beta(\mathrm{gr}^l_\alpha V) \dim (\mathrm{gr}^l_\alpha V)) + \sum_{l \in \mathbf{Q}} l\ \mu_\beta(\mathrm{gr}^l_\alpha W) \dim (\mathrm{gr}^l_\alpha W)\right) \\
&= c\left(-\int l\ d(\mu_\beta V^l_\alpha \dim V^l_\alpha) - \int l\ d(\mu_\beta W^l_\alpha \dim W^l_\alpha)\right)
\end{aligned}$$

Here we have an integral of a measure with finite support on the real line, and the minus sign occurs because α is a decreasing filtration of V.

$$\begin{aligned}
&= c[-\int l\ d((\mu_\beta V^l_\alpha - \mu_\beta V)\dim V^l_\alpha) - \int l\ d((\mu_\beta W^l_\alpha - \mu_\beta W)\dim W^l_\alpha) \\
&\quad +\mu_\alpha V\ \mu_\beta V \dim V + \mu_\alpha W\ \mu_\beta W \dim W] \\
&= c[\int (\mu_\beta V^l_\alpha - \mu_\beta V)\dim V^l_\alpha dl + \int (\mu_\beta W^l_\alpha - \mu_\beta W)\dim W^l_\alpha dl \\
&\quad +\mu_\alpha V\ \mu_\beta V \dim V + \mu_\alpha W\ \mu_\beta W \dim W],
\end{aligned}$$

by integration by parts. The Proposition follows since $\mu_\alpha V = \mu_\alpha W = 0$ by Proposition 1. QED.

4 Main theorem

We will state the theorem, give some convenient notation and a few variants of the theorem, and then prove the theorem.

Theorem 1 *Let k be a perfect field of characteristic $p > 0$, $K_0 = W(k) \otimes_{\mathbf{Z}} \mathbf{Q}$, and K a finite extension of K_0. Let V and W be weakly admissible filtered isocrystals over K. Then the filtered isocrystal $V \otimes W$ is weakly admissible.*

Here the filtration on $V \otimes W$ is given by $(V \otimes W)^i = \sum V^j \otimes W^{i-j}$, as in section 2, and φ is given on $V_0 \otimes W_0$ by $\varphi(v \otimes w) = \varphi v \otimes \varphi w$.

For any filtered isocrystal V, we consider two filtrations of the K-vector space V: the given one (V^i), which we call λ_1, and a filtration λ_2 indexed by $\mathbf{Q}$ which is defined by

$$V^i_{\lambda_2} = (\oplus_{l \leq -i}(V_0)_{\text{slope } l}) \otimes_{K_0} K \subset V.$$

We can formally define an "object" $\lambda = \lambda_1 + \lambda_2$; λ is not a filtration of V, but we can still define $\deg_\lambda V = \deg_{\lambda_1} V + \deg_{\lambda_2} V$ and $\mu_\lambda V = \deg_\lambda V / \dim V$. In these terms, a filtered isocrystal V is weakly admissible if $\mu_\lambda S \leq 0$ for all $S = S_0 \otimes_{K_0} K$, $S_0 \subset V_0$ a φ-invariant subspace, with equality for $S_0 = V_0$.

There is a slight generalization of weak admissibility which is sometimes useful. We say that a filtered isocrystal V is semistable of slope c if $\mu_\lambda V = c$ and $\mu_\lambda(S) \leq c$ for all sub-isocrystals $S_0 \subset V_0$. Thus weak admissibility is just semistability of slope 0. Faltings defined a Harder-Narasimhan filtration on filtered isocrystals so that the subquotients are semistable of various slopes [5].

Corollary 1 *The tensor product of semistable filtered isocrystals is semistable.*

Proof. This follows immediately from the theorem, the case of slope 0, since a filtered isocrystal is semistable of slope c if and only if shifting the filtration λ_1 down by c gives a weakly admissible filtered isocrystal. QED.

Remark. A few other variants of the theorem which can be proved by the same argument as the theorem itself are: we can replace the field $K_0 = W(k) \otimes_{\mathbf{Z}} \mathbf{Q}$ in the definition of an isocrystal by the power series field $k((t))$, with the automorphism σ being the Frobenius on k and the identity on t; and, instead of filtered isocrystals, we can consider vector spaces over an arbitrary field with a finite set of filtrations, as in [6] and [5].

Proof of Theorem 1. First, we observe that we can assume the finite extension field K of K_0 to be Galois over K_0. Indeed, weak admissibility of a filtered isocrystal V over K is just an inequality on all φ-invariant subspaces of V_0 over K_0, so this condition does not change if we replace K by a bigger finite extension L of K_0 and V by $V \otimes_K L$. After making such a change, we can assume from now on that K is Galois over K_0.

The equality statement, that $\mu_\lambda(V \otimes W) = 0$, is obvious from the corresponding statements for V and W. So we just have to prove that for every φ-invariant subspace $S_0 \subset V_0$, with $S = S_0 \otimes_{K_0} K$, we have $\mu_\lambda S \leq 0$.

If S were a sufficiently general K-linear subspace of $V \otimes W$, the inequality $\mu_\lambda S \leq 0$ would be obvious: $\mu_\lambda S$ can only be big if S has high-dimensional intersection with some of the subspaces $(V\otimes W)^i_{\lambda_1}$ or $(V\otimes W)^i_{\lambda_2}$ of $V\otimes W$. On the other hand, if S is a very special subspace of $V\otimes W$, say a decomposable subspace $S=S_1\otimes S_2$, then the inequality $\mu_\lambda S\leq 0$ is also easy to prove from the corresponding inequalities for S_1 and S_2. The difficulty is what to do if S is somewhere in the middle. The solution is to apply geometric invariant theory to give a sharp dichotomy between "general" subspaces and "special" subspaces of $V\otimes W$, in such a way that we get useful information in either case. This idea comes from Ramanan and Ramanathan's paper.

Namely, if $S=S_0\otimes K$ is a $GL(V)\times GL(W)$-semistable subspace of the K-vector space $V\otimes W$ (as defined in section 2), then S has smaller slope than $V\otimes W$ with respect to any filtrations of the K-vector spaces V and W. In particular

$$\mu_{\lambda_1}S\leq \mu_{\lambda_1}(V\otimes W)$$

and

$$\mu_{\lambda_2}S\leq \mu_{\lambda_2}(V\otimes W),$$

so that (adding these inequalities) $\mu_\lambda S\leq \mu_\lambda(V\otimes W)=0$.

So we just have to prove that $\mu_\lambda S\leq 0$ when S is not $GL(V)\times GL(W)$-semistable. By Proposition 1, to S is naturally associated a filtration $(V^i_{\alpha(S)},W^i_{\alpha(S)})$ of (V,W) (unique up to scaling) which maximizes

$$f(S,\alpha)=\frac{\mu S-\mu(V\otimes W)}{(\sum i^2 \dim \mathrm{gr}^i V+\sum i^2 \dim \mathrm{gr}^i W)^{1/2}}$$

as a function of a filtration α of the K-vector spaces (V,W). By scaling, we can arrange that Kempf's filtration $\alpha(S)$ is indexed by the integers.

The uniqueness of Kempf's filtration allows us to prove that the filtration $(V^i_{\alpha(S)},W^i_{\alpha(S)})$ associated to a K-linear subspace $S\subset V\otimes W$ has further structure if S does. Namely, since $V=V_0\otimes_{K_0}K$ and $W=W_0\otimes_{K_0}K$, the Galois group $Gal(K/K_0)$ acts on V, W, and $V\otimes_K W$, and $f(gS,g\alpha)=f(S,\alpha)$ for all K-linear subspaces $S\subset V\otimes W$, filtrations α of (V,W), and $g\in Gal(K/K_0)$; so the maximizer $\alpha(gS)$ associated to gS is $g(\alpha(S))$. Thus if $S=S_0\otimes K$ for some K_0-subspace $S_0\subset V_0\otimes W_0$ (which is equivalent to S being preserved by $Gal(K/K_0)$), then the associated filtration $\alpha(S)$ of (V,W) comes from a filtration (which I will call $\alpha(S_0)$) of (V_0,W_0).

Likewise, one has $f(\varphi S_0,\varphi\alpha)=f(S_0,\alpha)$ for all K_0-subspaces $S_0\subset V_0\otimes W_0$ and filtrations α of (V_0,W_0). (To define the function f, one first tensors these objects up to K.) Since S_0 and φS_0 are K_0-subspaces, by the previous paragraph we know that their associated filtrations $\alpha(S_0)$ and $\alpha(\varphi S_0)$ are defined over K_0; and the equality just stated then shows that $\alpha(\varphi S_0)=\varphi(\alpha(S_0))$. Thus, in the situation of interest, where $S_0\subset V_0\otimes W_0$ is a sub-isocrystal ($\varphi S_0=S_0$), Kempf's filtration $\alpha(S_0)$ is a filtration of (V_0,W_0) *by sub-isocrystals.*

To sum up, returning to the proof that $V\otimes W$ is weakly admissible: if the subspace $S=S_0\otimes_{K_0}K$ is not $GL(V)\times GL(W)$-semistable, we have associated to S a filtration $(V^i_{\alpha(S_0)},W^i_{\alpha(S_0)})$ of (V_0,W_0) by sub-isocrystals. This allows us to use our assumption that V and W are weakly admissible: we know that

$$\mu_\lambda V^l_{\alpha(S_0)}\leq \mu_\lambda V$$

and

$$\mu_\lambda W^l_{\alpha(S_0)}\leq \mu_\lambda W$$

for all $l \in \mathbf{Z}$. (Recall that λ is not a filtration but a "sum" of the two filtrations λ_1 and λ_2, on V, W, or $V \otimes W$.)

Finally, by Proposition 2, there is a constant $c > 0$ such that

$$\mu_\beta S - \mu_\beta(V \otimes W) \leq c \left(\int (\mu_\beta V^l_{\alpha(S_0)} - \mu_\beta V) \dim V^l_{\alpha(S_0)} dl + \int (\mu_\beta W^l_{\alpha(S_0)} - \mu_\beta W) \dim W^l_{\alpha(S_0)} dl \right).$$

for all filtrations β of the K-vector spaces (V, W). Using this inequality for $\beta = \lambda_1$ and $\beta = \lambda_2$ and adding the results, we find that

$$\mu_\lambda S - \mu_\lambda(V \otimes W) \leq c \left(\int (\mu_\lambda V^l_{\alpha(S_0)} - \mu_\lambda V) \dim V^l_{\alpha(S_0)} dl + \int (\mu_\lambda W^l_{\alpha(S_0)} - \mu_\lambda W) \dim W^l_{\alpha(S_0)} dl \right).$$

But the right hand side is ≤ 0 by the previous paragraph. So $\mu_\lambda S \leq \mu_\lambda(V \otimes W)$.

Thus, whether S is $GL(V) \times GL(W)$-semistable or not, we have proved that $\mu_\lambda S \leq \mu_\lambda(V \otimes W)$ for $S = S_0 \otimes_{K_0} K$, $S_0 =$ sub-isocrystal of $V_0 \otimes W_0$. That is, $V \otimes W$ is weakly admissible. QED.

5 Other categories

The same proof shows that, in addition to the category $MF^w_K(\varphi)$ of weakly admissible filtered isocrystals, some bigger abelian categories defined by Fontaine [8], $MF^w_K(\varphi, N)$ and $MF^w_{L/K}(\varphi, N)$, are likewise closed under tensor product, as Fontaine conjectured. We give the definitions below. Just as the de Rham cohomology of a variety with good reduction over a p-adic field K can be given the structure of an object of $MF^w_K(\varphi)$, the de Rham cohomology of a variety with semistable reduction can be given the structure of an object of $MF^w_K(\varphi, N)$, at least for $\dim X < (p-1)/2$ [10]. Here N stands for a nilpotent "logarithm of the monodromy" operator as in the theory of variations of Hodge structures. More generally, a variety over K which has semistable reduction over some Galois extension field L/K defines an object of $MF^w_{L/K}(\varphi, N)$ under the same dimension assumption. One expects that in fact every variety over K defines an object of $MF^w_{L/K}(\varphi, N)$ for some finite extension L/K and so also for L equal to the algebraic closure of K.

We define these categories. Again, let K be a field of characteristic 0, complete with respect to a discrete valuation, with residue field k perfect of characteristic $p > 0$. Let L be a Galois extension field of K, with residue field k_L and with Galois group $G_{L/K}$ over K, possibly infinite. Let L_0 be the maximal unramified extension of $K_0 = \mathrm{Frac}\, W(k)$ contained in L. Let σ denote the Frobenius automorphism of L_0 (that is, the unique automorphism inducing $x \mapsto x^p$ on k_L).

We define a $(\varphi, N, G_{L/K})$-*module* to be a finite-dimensional L_0-vector space V_0, together with

(1) a bijective, σ-linear ($\varphi(cx) = \sigma(c)\varphi(x)$, $c \in L_0$) endomorphism

$$\varphi : V_0 \to V_0,$$

(2) an L_0-linear endomorphism

$$N : V_0 \to V_0,$$

(3) and a semilinear ($g(cx) = g(c)g(x)$, $c \in L_0$, $g \in G_{L/K}$) action of $G_{L/K}$ on V_0, such that the isotropy group of each point in V_0 is an open subgroup of $G_{L/K}$.

We impose the following relations between these maps:

(a) we have $N\varphi = p\varphi N$;

(b) for every $g \in G_{L/K}$, we have $g\varphi = \varphi g$ and $gN = Ng$.

Condition (a) implies that N maps the subspace of φ-slope l into the subspace of φ-slope $l-1$; in particular, N is nilpotent.

The $(\varphi, N, G_{L/K})$-modules form a $\mathbf{Q}_p$-linear abelian category. The category of (φ, N)-modules, by definition, is just the special case of this category in which $L = K$ (so that the action of $G_{L/K}$ can be omitted from the definition). The tensor product $V_0 \otimes W_0$ of two $(\varphi, N, G_{L/K})$-modules is defined by the following operations on the L_0-vector space $V_0 \otimes_{L_0} W_0$:

$$\varphi(v \otimes w) = \varphi v \otimes \varphi w, \;\; N(v \otimes w) = Nv \otimes w + v \otimes Nw, \;\; g(v \otimes w) = gv \otimes gw.$$

This makes the $(\varphi, N, G_{L/K})$-modules into a tannakian category [8].

If V_0 is a $(\varphi, N, G_{L/K})$-module, the action of N (resp. $G_{L/K}$) is extended by linearity (resp. semi-linearity) to $V := V_0 \otimes_{L_0} L$. We define a *filtered* $(\varphi, N, G_{L/K})$-*module* to be a $(\varphi, N, G_{L/K})$-module V_0 together with a filtration (V^i) indexed by $i \in \mathbf{Z}$ of $V := V_0 \otimes_{L_0} L$ by L-linear subspaces invariant under $G_{L/K}$. As earlier in this paper, a filtration is by definition decreasing ($V^i \supset V^{i+1}$) and exhaustive ($\cup V^i = V$, $\cap V^i = 0$).

Let V be a filtered $(\varphi, N, G_{L/K})$-module. We define the Hodge number $\mu_{\lambda_1}(V)$ just using the filtration of V, as

$$\mu_{\lambda_1}(V) = \sum i \; \dim_L V^i / V^{i+1},$$

and the negative of the Newton number, $\mu_{\lambda_2}(V)$, just using the σ-linear endomorphism φ of V_0: if $V_0 = \oplus_{l \in \mathbf{Q}} (V_0)_l$ is the slope grading (see section 1), then we define

$$\mu_{\lambda_2}(V) = \sum_{l \in \mathbf{Q}} l \; \dim (V_0)_l.$$

A filtered $(\varphi, N, G_{L/K})$-module is defined to be *weakly admissible* if $\mu_\lambda(V) := \mu_{\lambda_1}(V) + \mu_{\lambda_2}(V)$ is 0, and for every sub-$(\varphi, N, G_{L/K})$-module $S_0 \subset V_0$, with $S := S_0 \otimes_{L_0} L$ given the induced filtration, we have $\mu_\lambda(S) \leq 0$. Let $MF^w_{L/K}(\varphi, N)$ denote the category of weakly admissible filtered $(\varphi, N, G_{L/K})$-modules. In the special case $L = K$, this is called the category $MF^w_K(\varphi, N)$ of weakly admissible filtered (φ, N)-modules.

Theorem 2 *The tensor product of two weakly admissible filtered $(\varphi, N, G_{L/K})$-modules is weakly admissible.*

Fontaine [8] conjectured this statement and mentioned that it implies that $MF^w_{L/K}(\varphi, N)$ is a tannakian category.

Proof of Theorem. The proof is the natural generalization of the case $N = 0$, $L = K$ considered in Theorem 1. Let V and W be weakly admissible filtered $(\varphi, N, G_{L/K})$-modules. Let $S_0 \subset V_0 \otimes W_0$ be a sub-$(\varphi, N, G_{L/K})$-module, and let $S = S_0 \otimes_{L_0} L$, which is a filtered $(\varphi, N, G_{L/K})$-module; we need to show that

$\mu_\lambda(S) \leq 0$. This is true if $S \subset V \otimes W$ is $GL(V) \times GL(W)$-semistable, by the same argument as in the proof of Theorem 1; this case does not use the assumption that S_0 is invariant under N and $G_{L/K}$, or the weak admissibility of V and W.

If S is not $GL(V) \times GL(W)$-semistable, then Proposition 1 shows that S determines a filtration $(V^i_{\alpha(S)}, W^i_{\alpha(S)})$ of V and W by L-linear subspaces, and the desired inequality will be proved as before once we know that the subspaces $V^i_{\alpha(S)} \subset V$, $W^i_{\alpha(S)} \subset W$ come from sub-$(\varphi, N, G_{L/K})$-modules of V_0 and W_0, given that S comes from a sub-$(\varphi, N, G_{L/K})$-module of $V_0 \otimes W_0$. As before, this follows from the uniqueness of Kempf's filtration, as follows. If we briefly let L' be an extension of L which is Galois over L_0, then $S \otimes_L L'$ is invariant under the action of the group $Gal(L'/L_0)$, and so Kempf's filtration associated to $S \otimes_L L'$ is also, so that it comes from a filtration which we call $\alpha(S_0)$ of the L_0-vector spaces (V_0, W_0). Kempf's filtration associated to S (as opposed to $S \otimes_L L'$) comes from this same filtration $\alpha(S_0)$, since this filtration maximizes Kempf's function (Proposition 1) even among the larger class of all filtrations defined over L'. Since S_0 is mapped onto itself by the endomorphism φ, the group $\{e^{tN} : t \in \mathbf{Q}\}$, and the group $G_{L/K}$ (all acting on $V_0 \otimes W_0$), so is the filtration $\alpha(S_0)$. QED

As it happens, this proof does not use the compatibilities between φ, N, and $G_{L/K}$ in the definition of a $(\varphi, N, G_{L/K})$-module, except to know that N is nilpotent so that the group $\{e^{tN} : t \in \mathbf{Q}\}$ makes sense.

6 A lemma

Now we turn to the second subject of this paper, Rapoport and Zink's conjectured characterization of weak admissibility for filtered isocrystals with G-structure in terms of geometric invariant theory ([19], 1.51), which is formulated and proved in section 8. In this section we prove it for $G = GL(n)$, where it is more or less obvious: it is a matter of going through the various definitions.

Definition. Let V be a vector space over a field K with two filtrations α and β (as defined at the beginning of section 2). We define the "inner product" $\langle \alpha, \beta \rangle$ to be

$$\langle \alpha, \beta \rangle = \sum_{k,l \in \mathbf{Q}} \dim \mathrm{gr}^k_\alpha \mathrm{gr}^l_\beta V.$$

Given a filtration κ of a vector space V, let $P(\kappa)$ be the corresponding parabolic subgroup of $GL(V)$, that is, the subgroup of $GL(V)$ which preserves the filtration κ.

Definition. Let L_κ be the character $\otimes\mathbf{Q}$ of $P(\kappa)$ defined by

$$L_\kappa = \otimes_k (\det \mathrm{gr}^k_\kappa V)^{\otimes -k}.$$

Of course, if the filtration κ is indexed by the integers, then L_κ is actually a character of $P(\kappa)$; in general, it is an element of $X^*(P(\kappa)) \otimes \mathbf{Q}$. As a result, it defines an element of $\mathrm{Pic}^G L(V)(\mathrm{Flag}(V)) \otimes \mathbf{Q}$, where $\mathrm{Flag}(V)$ is the partial flag variety of all filtrations of V with the same $\dim \mathrm{gr}^i V$ as κ, since we can identify $\mathrm{Flag}(V)$ with $GL(V)/P(\kappa)$. An element of $\mathrm{Pic}^G \otimes \mathbf{Q}$, which we informally call a "G-line bundle $\otimes\mathbf{Q}$," is perfectly adequate for defining semistability (as in the following lemma), since a point x in a G-variety X is semistable with respect to a G-line bundle L

if and only if it is semistable with respect to $L^{\otimes n}$, $n > 0$. (Consequently we could avoid all mention of "line bundles $\otimes\mathbf{Q}$" if we wanted to, but I find them convenient.)

Now we can state the goal of this section, the characterization of weak admissibility conjectured by Rapoport and Zink in the case of $GL(n)$.

Lemma 5 *Let K be a complete, discretely valued field with algebraically closed residue field k, and let V be a filtered isocrystal over K. Let λ_1, λ_2 be the filtrations of V defined by: λ_1 = given filtration of V, and*

$$V^i_{\lambda_2} = \oplus_{l\leq -i, l\in\mathbf{Q}}((V_0)_{slope\ l}) \otimes_{K_0} K.$$

Let Flag(V) be the flag variety of filtrations of V of the same type as λ_1 (same dim $gr^i V$). Let J be the group over $\mathbf{Q}_p$ such that $J(\mathbf{Q}_p)$ is the group of automorphisms of the isocrystal (V_0, φ) ([19], Proposition 1.12); since J_{K_0} is a subgroup of $GL(V_0)$, J_{K_0} acts on the flag variety Flag(V_0).

Then V is weakly admissible $\iff$ the point $\lambda_1 \in$ Flag(V) is semistable with respect to all one-parameter subgroups $G_m \to J$ defined over $\mathbf{Q}_p$ and the J_{K_0}-line bundle $\otimes\mathbf{Q}$, $L_{\lambda_1} \otimes L_{\lambda_2}$, on Flag$(V_0)$.

In terms of Mumford's numerical invariant $\mu(x, \alpha, L)$ of a point x in a projective G-variety X, a one-parameter subgroup α of G, and a G-line bundle L on X ([15], p. 49), the second condition in the above equivalence means that $\mu(\lambda_1, \alpha, L_{\lambda_1} \otimes L_{\lambda_2}) \geq 0$ for all one-parameter subgroups α of J defined over $\mathbf{Q}_p$. Here L_{λ_1} is an ample line bundle on Flag(V), and L_{λ_2} is a trivial line bundle on which J_{K_0} acts by a character $\otimes\mathbf{Q}$, using that $J_{K_0} \subset P(\lambda_2) \subset GL(V_0)$.

Proof. By definition, V is weakly admissible if and only if $\mu_\lambda V = 0$ and $\mu_\lambda S \leq \mu_\lambda V$ for all $S = S_0 \otimes_{K_0} K$, S_0 = sub-isocrystal of V_0. We will briefly say that such a subspace S is a "sub-isocrystal of V." It follows easily that V is weakly admissible if and only if $\mu_\lambda V = 0$ and, for all filtrations α of V by sub-isocrystals, one has

$$\int (\mu_\lambda V^l_\alpha - \mu_\lambda V) \dim V^l_\alpha \, dl \leq 0.$$

This is the integral of a piecewise constant function with compact support on the real line.

A further reformulation of weak admissibility is that V is weakly admissible if and only if for all filtrations α of V by sub-isocrystals, one has $\langle\alpha, \lambda\rangle := \langle\alpha, \lambda_1\rangle + \langle\alpha, \lambda_2\rangle \leq 0$. To see this, we use the integration by parts from the proof of Proposition 2 to rewrite $\langle\alpha, \lambda\rangle$ as follows.

$$\langle\alpha, \lambda\rangle = \int (\mu_\lambda V^l_\alpha - \mu_\lambda V) \dim V^l_\alpha \, dl + \mu_\lambda V \mu_\alpha V \dim V.$$

If V is weakly admissible, then $\mu_\lambda V = 0$, so the right term is 0 and $\langle\alpha, \lambda\rangle \leq 0$ follows from the previous paragraph's inequality. Conversely, if $\langle\alpha, \lambda\rangle \leq 0$ for all filtrations α of V by sub-isocrystals, then we can apply this in particular to the trivial filtration $V^i_\alpha = V$ for $i \leq j$, $V^i_\alpha = 0$ for $i > j$, and the inequality $\langle\alpha, \lambda\rangle \leq 0$ means that

$$j \cdot \mu_\lambda V \leq 0,$$

where $j \in \mathbf{Q}$ is arbitrary. It follows that $\mu_\lambda = 0$. So the assumption that $\langle \alpha, \lambda \rangle \leq 0$ for all filtrations α of V by sub-isocrystals just amounts to the previous paragraph's inequality, and so V is weakly admissible.

Finally, one-parameter subgroups $G_m \to J$ defined over $\mathbf{Q}_p$ are in one-to-one correspondence with splittings of V_0 as a direct sum of sub-isocrystals, indexed by $\mathbf{Z}$. Since the category of isocrystals is semisimple, every filtration of V_0 by sub-isocrystals, indexed by $\mathbf{Z}$, comes from some one-parameter subgroup of J defined over $\mathbf{Q}_p$.

Thus V is weakly admissible if and only if $\langle \alpha, \lambda \rangle \leq 0$ for all one-parameter subgroups $\alpha : G_m \to J$ defined over $\mathbf{Q}_p$. Now the proof will be complete after the following sub-lemma. QED.

Lemma 6 *Let λ_2 be a filtration of a vector space V over a field K, and let $\alpha : G_m \to GL(V)$ be a one-parameter subgroup which preserves λ_2. Let Flag(V) be the flag variety of filtrations λ_1 of V of a given type (dim $gr^i V$ fixed). Then Mumford's numerical invariant of a point $\lambda_1 \in$ Flag(V) with respect to the one-parameter subgroup α and the G_m-line bundle $L_{\lambda_1} \otimes L_{\lambda_2}$ is given by*

$$\mu(\lambda_1, \alpha, L_{\lambda_1} \otimes L_{\lambda_2}) = -\langle \alpha, \lambda \rangle,$$

where by definition $\langle \alpha, \lambda \rangle = \langle \alpha, \lambda_1 \rangle + \langle \alpha, \lambda_2 \rangle$.

Proof. Mumford's numerical invariant is additive in the G_m-line bundle. Since L_{λ_1} and L_{λ_2} are defined as tensor products of simpler line bundles, it suffices to check that

$$\mu(\lambda_1, \alpha, \det \mathrm{gr}^l_{\lambda_1} V) = \sum_k k \dim \mathrm{gr}^k_\alpha \mathrm{gr}^l_{\lambda_1} V$$

and

$$\mu(\lambda_1, \alpha, \det \mathrm{gr}^l_{\lambda_2} V) = \sum_k k \dim \mathrm{gr}^k_\alpha \mathrm{gr}^l_{\lambda_2} V.$$

The second equality is trivial to check, since $\det \mathrm{gr}^l_{\lambda_2} V$ is just a character of G_m, independent of the point $\lambda_1 \in \mathrm{Flag}(V)$. The first equality follows from the statement that $\mu(\lambda_1, \alpha, \det V^l_{\lambda_1})$ is equal to $\sum_k k \dim \mathrm{gr}^k_\alpha (V^l_{\lambda_1}) = \deg_\alpha V^l_{\lambda_1}$ for all l; since the G_m-line bundle $\det V^l_{\lambda_1}$ on the flag variety is pulled back from a Grassmannian, that statement follows from Mumford's calculation of his invariant on a Grassmannian (see the proof of Lemma 2, where the line bundle $(\det S)^*$ is used instead on $\det S$). QED.

7 Invariant inner products

In order to state the next section's characterization of weak admissibility, we need the notion of an invariant inner product (or just "inner product" for short) on a reductive group G over a field k. This means a positive definite inner product on $X_*(T) \otimes \mathbf{Q}$ for every maximal torus T in $G_{\overline{k}}$, $\overline{k}$ = separable closure of k, such that these inner products are preserved by conjugation by $G(\overline{k})$ and by the action of $Gal(\overline{k}/k)$. (Here $X_*(T)$ is the group of one-parameter subgroups $G_m \to T$.) An equivalent notion was used by Kempf [11], pp. 312-313; also, such inner products are familiar in the theory of buildings, because they give $Gal(\overline{k}/k)$-invariant metrics on the spherical building of G over $\overline{k}$ (see Mumford [15], p. 59, for example). In this section we will describe the set of inner products on a reductive group, as we will need for the proof of Theorem 3 in section 8. We will also explain how an inner product together with a one-parameter subgroup of G determine a character of an associated parabolic subgroup, which we need even to state Theorem 3.

We begin by giving a more efficient description of the inner products on G, in terms of just one maximal torus $T \subset G_{\overline{k}}$. Choose a Borel subgroup B containing T; then one can define a natural action of the Galois group $Gal(\overline{k}/k)$ on $X_*(T)$: given $\gamma \in Gal(\overline{k}/k)$, there is a $g \in G(\overline{k})$ such that $g \cdot^{\gamma} T \cdot g^{-1} = T$ and $g \cdot^{\gamma} B \cdot g^{-1} = B$, and the resulting automorphism of $X_*(T)$ is independent of the choice of g ([2], 1.3). The Galois group acts through a finite quotient on $X_*(T)$, and invariant inner products on G are in one-to-one correspondence with $Gal(\overline{k}/k) \ltimes W$-invariant inner products on $X_*(T) \otimes \mathbf{Q}$, where $W = N(T)/T$ is the Weyl group. In particular, this makes it clear that every connected reductive group has at least one invariant inner product.

If G is semisimple, there is a natural choice of an inner product, the Killing form

$$\langle \alpha, \beta \rangle = \sum_{\text{roots } \chi \text{ in } X_*(T)} \chi(\alpha)\chi(\beta) \in \mathbf{Z},$$

$\alpha, \beta \in X_*(T)$. But there is in general no natural choice of inner product when G is a torus.

To describe the set of inner products on G, we need to recall some of the structure theory of reductive groups. Define a connected algebraic group G over a field k to be k-simple if it is not abelian, and all closed normal subgroups are either finite or the whole group. Also, say that an algebraic group G is the almost direct product of subgroups $G_1, \ldots, G_n$ if the product map $G_1 \times \cdots \times G_n \to G$ is an isogeny, that is, a surjective homomorphism with finite kernel. By Borel-Tits [3], p. 64, every connected reductive group G over a field k is the almost direct product of the identity component of its center (which is a torus) with its derived group (which is a connected semisimple group). And by [3], p. 70, every semisimple group over k is the almost direct product of its k-simple normal subgroups. If G is a k-simple group, then applying this last result to $G_{\overline{k}}$, where $\overline{k}$ is the separable closure of k, we find that $G_{\overline{k}}$ is the almost direct product of its simple normal subgroups, and (since G is k-simple) the Galois group $Gal(\overline{k}/k)$ acts transitively on the set of simple normal subgroups of $G_{\overline{k}}$.

Lemma 7 *(1) Every inner product on a reductive group G over a field k is the orthogonal direct sum of inner products on the identity component of the center (a torus) and on the k-simple normal subgroups of G.*

(2) Every inner product on a k-simple group is a positive rational multiple of the Killing form.

Proof. (1) Over a separably closed field, all reductive groups are split, so they are classified by root systems just as over $\mathbf{C}$. The Weyl group of a simple group G over $\overline{k}$ acts by a nontrivial irreducible representation on $X_*(T) \otimes \mathbf{Q}$, for a maximal torus T: it is nontrivial because G is not abelian, and it is irreducible because a finite group generated by reflections which is reducible as an abstract representation (over $\mathbf{Q}$, or even over $\mathbf{C}$) is reducible as a group generated by reflections, as one easily checks. It follows that over $\overline{k}$, every invariant (that is, here, just Weyl-invariant) inner product on a reductive group is the orthogonal direct sum of invariant inner products on the identity component of the center (a torus) and on the simple normal subgroups. This statement over $\overline{k}$ is enough to imply statement (1) over k.

(2) Let G be a k-simple group, and let $H_{\overline{k}}$ be a $\overline{k}$-simple factor of G, with maximal torus $T \subset H_{\overline{k}}$. Since the Weyl group of H acts irreducibly on $X_*(T) \otimes \mathbf{Q}$, there is a unique invariant inner product on $H_{\overline{k}}$ up to positive rational scalars. By part (a), every inner product on $G_{\overline{k}}$ is the orthogonal direct sum of inner products on the $\overline{k}$-simple factors of G. Since the Galois group $Gal(\overline{k}/k)$ acts transitively on the $\overline{k}$-simple factors of G, there is a unique inner product on G up to positive rational scalars. QED.

Finally, here is the definition we need to state Theorem 3 in the next section. We follow Ramanan and Ramanathan [17], Remark 1.11. Let G be a connected reductive group over a field k, and fix an invariant inner product on G. For any one-parameter subgroup $\kappa : G_m \to G$, let $P(\kappa) \subset G$ be the associated parabolic subgroup defined by Mumford [15], p. 55. (Example: For $G = GL(V)$, a one-parameter subgroup κ is equivalent to a grading of V by the integers, and $P(\kappa)$ is the subgroup of $GL(V)$ which preserves the associated decreasing filtration of V.) Every maximal torus $\overline{T}$ in $P(\kappa)/U(\kappa)$ ($U(\kappa)$ = unipotent radical of $P(\kappa)$) is the isomorphic image of a maximal torus T of G contained in $P(\kappa)$, so the inner product on G gives one on $P(\kappa)/U(\kappa)$. The one-parameter subgroup κ maps into the center of $P(\kappa)/U(\kappa)$, which implies that the dual of κ with respect to the inner product, an element of $X^*(\overline{T}) \otimes \mathbf{Q}$, actually extends to a character $\otimes \mathbf{Q}$ of $P(\kappa)/U(\kappa)$. (We are using that the center of $P(\kappa)/U(\kappa)$ is orthogonal to its derived group, thanks to Lemma 7.)

Definition. Let $L_\kappa \in X^*(P(\kappa)) \otimes \mathbf{Q}$ be the negative of the dual of κ.

The point of the sign here is that the associated G-line bundle $\otimes \mathbf{Q}$, $L_\kappa \in \mathrm{Pic}^G(G/P(\kappa)) \otimes \mathbf{Q}$, is ample.

Let $D = \overset{\lim}{\leftarrow} G_m$ be the pro-torus over k with character group $X^*(D) = \mathbf{Q}$. Then we call a homomorphism $\kappa : D \to G$ a one-parameter subgroup $\otimes \mathbf{Q}$. (Example: One-parameter subgroups $\otimes \mathbf{Q}$ of $GL(V)$ correspond to gradings of V indexed by the rational numbers.) It is clear that the definition of $L_\kappa \in X^*(P(\kappa)) \otimes \mathbf{Q}$ extends to one-parameter subgroups $\otimes \mathbf{Q}$.

8 Filtered isocrystals with G-structure

In this final section, we define filtered isocrystals with G-structure, for a connected reductive group G over $\mathbf{Q}_p$, and we characterize weak admissibility for such objects in terms of geometric invariant theory. This characterization was conjectured by Rapoport and Zink ([19], 1.51). It is included in this paper because the proof is closely analogous to the proof of the tensor product theorem.

Let k be an algebraically closed field of characteristic $p > 0$, and let K_0 be the quotient field of the ring of Witt vectors $W(k)$. The most natural definition of an isocrystal with G-structure over K_0 is that it is an exact faithful tensor functor into the category of isocrystals over K_0,

$$\mathrm{Rep}_{\mathbf{Q}_p} G \to \mathrm{Isoc}(K_0),$$

as suggested by de Jong. For example, an isocrystal with $GL(n, \mathbf{Q}_p)$-structure is equivalent to an n-dimensional isocrystal, an isocrystal with $O(n)$-structure is equivalent to an n-dimensional isocrystal V_0 with a nondegenerate symmetric form $V_0 \otimes V_0 \to K_0$ which is a map of isocrystals, and so on (see Rapoport and Richartz [18]). Under our assumptions (G connected, the residue field k algebraically closed), [18] observes using a theorem of Steinberg's that every isocrystal with G-structure in the above sense can be obtained from some element $b \in G(K_0)$ by the following construction: to a given b, we associate the functor

$$\mathrm{Rep}_{\mathbf{Q}_p} G \to \mathrm{Isoc}(K_0)$$

which sends a representation V_{00} to the K_0-vector space $V_{00} \otimes_{\mathbf{Q}_p} K_0$ together with the σ-linear bijection $\varphi := b(1 \otimes \sigma)$. Two elements $b \in G(K_0)$ define isomorphic tensor functors if and only if they are σ-conjugate ($b \sim gb\sigma(g)^{-1}$, $g \in G(K_0)$).

Now suppose we are given a one-parameter subgroup $\lambda_1 : G_m \to G$ defined over a finite extension K of K_0, and an element $b \in G(K_0)$. Then to each $\mathbf{Q}_p$-representation V_{00} of G is associated a filtered isocrystal V, with $V_0 := V_{00} \otimes_{\mathbf{Q}_p} K_0$, $V := V_0 \otimes_{K_0} K$, $\varphi = b(1 \otimes \sigma)$ on V_0, and filtration on V given by $\lambda_1 : G_m \to G_K \to GL(V)$.

Following Rapoport and Zink [19], we say that a pair (λ_1, b) is weakly admissible if the filtered isocrystal V is weakly admissible for all $\mathbf{Q}_p$-representations of G. It is enough to check this for a single faithful representation of G, as one deduces from the theorem that tensor products of weakly admissibles are weakly admissible ([19], 1.18).

Fix a conjugacy class of one-parameter subgroups $\lambda_1 : G_m \to G$ which are defined over a fixed algebraic closure $\overline{K}_0$ of K_0. Let E be the field of definition of the conjugacy class: it is a finite extension of $\mathbf{Q}_p$ contained in $\overline{K}_0$. Two one-parameter subgroups $\lambda_1 : G_m \to G$ are said to give the same filtration of G if they define the same filtration of every representation of G. The set of filtrations λ_1 in the given geometric conjugacy class (or, as we call it, the set of filtrations λ_1 of the given *type*) which are defined over an extension field K of E form the K-points of a projective variety F over E which is a homogeneous space for G_E. (It may have no E-rational points.) For any such λ_1, we get an identification of this homogeneous space with $G/P(\lambda_1)$, $P(\lambda_1)$ being the parabolic subgroup associated to λ_1 by Mumford, [15], p. 55.

Rapoport and Zink define a group J over $\mathbf{Q}_p$ such that $J(\mathbf{Q}_p)$ is the subgroup of $G(K_0)$ which is fixed under conjugation by $b\sigma$ in the semidirect product $G(K_0) \rtimes \langle\sigma\rangle$ ([19], 1.12). Equivalently, J is the automorphism group of the isocrystal with G-structure defined by b. There is a natural homomorphism $J_{K_0} \to G_{K_0}$, and so J_E acts on F_E.

The tensor functor $\mathrm{Rep}_{\mathbf{Q}_p} G \to \mathrm{Isoc}(K_0)$ associated to b determines a functor from $\mathrm{Rep}_{\mathbf{Q}_p} G$ into graded vector spaces, by considering the slope grading of isocrystals, and thus a one-parameter subgroup of G defined over K_0. (More precisely, since isocrystals are in general graded by $\mathbf{Q}$, not $\mathbf{Z}$, we get naturally a homomorphism from the pro-torus $D := \overset{\lim}{\leftarrow} G_m$ with character group $\mathbf{Q}$ into G, which we call a one-parameter subgroup $\otimes\mathbf{Q}$ of G.) Let λ_2 be the inverse of this one-parameter subgroup $\otimes\mathbf{Q}$ of G; this is parallel to the definition of λ_2 at the beginning of section 4.

We now explain how a choice of invariant inner product on G, as defined in section 7, determines line bundles $\otimes\mathbf{Q}$, L_{λ_1} and L_{λ_2}, on the flag variety F of filtrations of G of the same type as λ_1. By the end of section 7, an inner product on G determines characters $\otimes\mathbf{Q}$ associated to λ_1 and λ_2: $L_{\lambda_1} \in X^*(P(\lambda_1)) \otimes \mathbf{Q}$ and $L_{\lambda_2} \in X^*(P(\lambda_2)) \otimes \mathbf{Q}$. Thus L_{λ_1} gives a G_K-line bundle $\otimes\mathbf{Q}$ on the flag variety $F_K = G_K/P(\lambda_1)$; since $J_{K_0} \subset G_{K_0}$, we can view L_{λ_1} as a J_K-line bundle $\otimes\mathbf{Q}$ on F_K. It is ample. Also, the group J of automorphisms of the given G-isocrystal preserves the slope grading, hence also the filtration λ_2, so that $J_{K_0} \subset P(\lambda_2)$; so L_{λ_2} gives a character $\otimes\mathbf{Q}$ of J_{K_0}, which we view as a J_{K_0}-line bundle $\otimes\mathbf{Q}$ on F_K (the trivial line bundle, with J_{K_0} acting by this character).

Thus $L_{\lambda_1} \otimes L_{\lambda_2}$ is a J_K-equivariant line bundle $\otimes\mathbf{Q}$ on F over K, depending on $b \in G(K_0)$, the conjugacy class of λ_1's, and an invariant inner product on G.

Now at last we can state the desired characterization of weak admissibility for filtered isocrystals with G-structure.

Theorem 3 *Let G be a connected reductive group over $\mathbf{Q}_p$. Fix an invariant inner product on G, as defined in section 7. Let b be an element of $G(K_0)$ and let $\lambda_1 : G_m \to G$ be a one-parameter subgroup defined over a finite extension K of K_0.*

Then (λ_1, b) is weakly admissible $\iff$ the point $\lambda_1 \in F(K)$ is semistable with respect to all one-parameter subgroups α of J defined over $\mathbf{Q}_p$, for the J_K-line bundle $\otimes\mathbf{Q}$, $L_{\lambda_1} \otimes L_{\lambda_2}$, on F.

The second of these two equivalent conditions can be stated in terms of Mumford's numerical invariant: $\mu(\lambda_1, \alpha, L_{\lambda_1} \otimes L_{\lambda_2}) \geq 0$ for all one-parameter subgroups α of J defined over $\mathbf{Q}_p$.

Proof. The first condition does not depend on a choice of inner product on G, while the second one a priori does, since the line bundle $L_{\lambda_1} \otimes L_{\lambda_2}$ depends on the inner product. We will first prove the equivalence for a particular inner product on G, and then check that the second condition is independent of the inner product.

Let V_{00} be a faithful representation of G over $\mathbf{Q}_p$. There is an obvious invariant inner product on $GL(V_{00})$, defined in terms of a basis for V_{00} as the inner product on $X_*(T) \otimes \mathbf{Q}$, for the diagonal maximal torus T, for which the n obvious one-parameter subgroups $G_m \to T$ are orthonormal. This restricts to give an invariant inner product on G, and we will now prove the equivalence in the theorem for this inner product.

The representation $G \hookrightarrow GL(V_{00})$ determines an imbedding of the homogeneous space F of filtrations $\lambda_1 : G_m \to G$ of a given type into the flag variety of filtrations $\lambda_1 : G_m \to GL(V)$ of a given type,

$$F \hookrightarrow \mathrm{Flag}(V_{00} \otimes_{\mathbf{Q}_p} K).$$

The imbedding is G_K-equivariant. The group J of automorphisms of the given G-isocrystal satisfies $J_{K_0} = G_{K_0} \cap J(GL)_{K_0} \subset GL(V_{00} \otimes K_0)$. In particular, it follows that the imbedding

$$F_K \hookrightarrow \mathrm{Flag}(V_{00} \otimes K)$$

is J_K-equivariant.

A one-parameter subgroup $\kappa : G_m \to G \subset GL(V_{00})$, say defined over some extension field of $\mathbf{Q}_p$, determines characters L_κ of both $P(\kappa) \subset G$ and $P_{GL(V)}(\kappa) \subset GL(V)$, where $V = V_{00} \otimes K$. Here $P(\kappa) = G \cap P_{GL(V)}(\kappa)$ by Mumford [15], pp. 55-56, and with our choice of inner product on G, it is easy to check that the character L_κ on $P(\kappa)$ is just the restriction on the character L_κ on $P_{GL(V)}(\kappa)$. Also, we can compute that L_κ on $P_{GL(V)}(\kappa)$ is the character

$$L_\kappa = \otimes_k (\det \operatorname{gr}_\kappa^k V)^{\otimes -k}.$$

By the remark after our definition of weak admissibility of (λ_1, b), (λ_1, b) is weakly admissible if and only if the filtered isocrystal $V := (V_{00} \otimes_{\mathbf{Q}_p} K, b(1 \otimes \sigma), \lambda_1)$ is weakly admissible. By Lemma 5, V is weakly admissible if and only if $\lambda_1 \in \operatorname{Flag}(V)$ is semistable with respect to all one-parameter subgroups $G_m \to J(GL)$ defined over $\mathbf{Q}_p$ and the $J_K(GL)$-line bundle $L_{\lambda_1} \otimes L_{\lambda_2}$. Clearly this implies that λ_1 is semistable with respect to one-parameter subgroups of $J(GL)$ which are also contained in $J_{K_0} = J(GL(V_0)) \cap G_{K_0}$, and we just have to prove the converse.

We can prove this using the same argument which shows that the tensor product of weakly admissible filtered isocrystals is weakly admissible. Probably this theorem could be deduced from the tensor product theorem instead.

Thus, suppose that $\lambda_1 \in F(K)$ is semistable with respect to one-parameter subgroups $G_m \to J$ defined over $\mathbf{Q}_p$ and the line bundle $L_{\lambda_1} \otimes L_{\lambda_2}$. We will show that V is weakly admissible. That is, by the proof of Lemma 5, we need to show that for all filtrations α of V_0 (and thus of V) by sub-isocrystals, one has $\langle \alpha, \lambda \rangle \le 0$. So let α be such a filtration, represented by a K_0-rational point in a different flag variety, $\operatorname{Flag}_\alpha(V_0)$.

We ask whether α is semistable with respect to the action of G_K on $\operatorname{Flag}_\alpha(V_0 \otimes_{K_0} K)$ and the ample G_K-line bundle

$$L_\alpha = \otimes_i (\det \operatorname{gr}_\alpha^i V)^{\otimes -i}.$$

Equivalently, by Lemma 6 applied in the case of trivial λ_2, we are asking whether $\langle \alpha, \beta \rangle \le 0$ for all one-parameter subgroups $\beta : G_m \to G_K$ (viewed as giving filtrations on V, which is a representation of G_K).

If this is true, then in particular $\langle \alpha, \lambda_1 \rangle \le 0$ and $\langle \alpha, \lambda_2 \rangle \le 0$, since the filtrations λ_1 and λ_2 can be represented by one-parameter subgroups $\lambda_1 : G_m \to G_K$ and $\lambda_2 : G_m \to G_{K_0}$. (More precisely, some positive multiple of λ_2 is represented by such a one-parameter subgroup, which is enough to prove the inequality $\langle \alpha, \lambda_2 \rangle \le 0$.) So $\langle \alpha, \lambda \rangle \le 0$, which is the inequality we want.

The alternative is that α is not G_K-semistable. By Kempf [11], to the non-semistable point α is associated a set of one-parameter subgroups $\beta : G_m \to G_K$ such that the resulting filtration β on K-representations of G is unique up to scaling. Namely, β is the filtration of $\operatorname{Rep}(G_K)$ which maximizes

$$\langle \alpha, \beta \rangle / |\beta|,$$

where $\langle \alpha, \beta \rangle$ and $|\beta|$ are defined using the inner product on G we are using, that is, they are defined by thinking of α and β as filtrations of the particular representation V of G_K.

We now recall that the filtration α of V comes from a filtration of V_0 by sub-isocrystals. By the uniqueness of Kempf's filtration β of $\mathrm{Rep}(G_K)$, it follows as in the proof of the tensor product theorem (Theorem 1) that β is represented by a one-parameter subgroup $\beta : G_m \to J$ over $\mathbf{Q}_p$.

We can now use our assumption that $\lambda_1 \in F(K)$ is semistable with respect to one-parameter subgroups $G_m \to J$ over $\mathbf{Q}_p$ and the line bundle $L_{\lambda_1} \otimes L_{\lambda_2}$. By Lemma 6, this assumption implies that

$$\langle \beta, \lambda \rangle \leq 0.$$

Finally, we apply Ramanan and Ramanathan's Proposition 1.12 [17] (generalizing Proposition 2 in this paper) which gives a useful property of Kempf's filtration β of $\mathrm{Rep}(G_K)$ associated to $\alpha \in \mathrm{Flag}_\alpha(V)$: there is a constant $c > 0$ such that for all filtrations γ of $\mathrm{Rep}(G_K)$, one has

$$\langle \alpha, \gamma \rangle \leq c \langle \beta, \gamma \rangle.$$

Applying this to $\gamma = \lambda_1$ and $\gamma = \lambda_2$ and adding the results shows that

$$\langle \alpha, \lambda \rangle \leq c \langle \beta, \lambda \rangle$$

which is ≤ 0 by the previous paragraph. Thus for any filtration α of V_0 by sub-isocrystals, we have proved that $\langle \alpha, \lambda \rangle \leq 0$, as we needed.

Thus, we have proved the theorem for the inner product on G coming from a faithful representation of G. We now have to prove the theorem for an arbitrary inner product on G.

By Lemma 7, any two inner products on a $\mathbf{Q}_p$-simple group G differ by a positive rational constant. As a result, the theorem is true for all invariant inner products on a $\mathbf{Q}_p$-simple group G, since we have checked it for the inner product coming from a faithful representation of G, and the second condition in the theorem clearly does not change when the inner product changes by a positive scalar factor. (The line bundles L_{λ_1} and L_{λ_2} are multiplied by the same positive number.)

Also, we can prove the theorem directly for an arbitrary invariant inner product when the given group is a torus. By Rapoport and Zink [19], Proposition 1.21, for a torus T over $\mathbf{Q}_p$, a pair (λ_1, b) is weakly admissible if and only if $\lambda_1 + \lambda_2 \in X_*(T) \otimes \mathbf{Q}$ is orthogonal to all $\mathbf{Q}_p$-rational characters of T. (This notion of "orthogonality" does not depend on a choice of inner product on $X_*(T) \otimes \mathbf{Q}$, since the character group $X^*(T)$ is naturally dual to $X_*(T)$.) On the other hand, the second condition in our theorem says that for all one-parameter subgroups $\alpha : G_m \to J$ defined over $\mathbf{Q}_p$, we have $\langle \alpha, \lambda \rangle := \langle \alpha, \lambda_1 \rangle + \langle \alpha, \lambda_2 \rangle \leq 0$, with respect to a given invariant inner product on $X_*(T) \otimes \mathbf{Q}$. Since T is abelian, the definition of J shows that J is canonically isomorphic to T over $\mathbf{Q}_p$. So the second condition in the theorem is that $\langle \alpha, \lambda_1 + \lambda_2 \rangle \leq 0$ for all $\alpha \in X_*(T)^{Gal(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)}$. By applying this to α and $-\alpha$, we see that it is equivalent to $\lambda_1 + \lambda_2 \in X_*(T) \otimes \mathbf{Q}$ being orthogonal to the $Gal(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$-invariants in $X_*(T)$. Now we have assumed that our inner product is invariant, which for a torus just means $Gal(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$-invariant, so that for an element of $X_*(T) \otimes \mathbf{Q}$ to be orthogonal to the Galois invariants in $X_*(T)$ is equivalent to its being orthogonal to the Galois invariants in $X^*(T)$, which as we noted above is equivalent to (λ_1, b) being weakly admissible. Thus we have proved the theorem for arbitrary inner products when the given group is a torus.

It follows that the theorem is true for arbitrary inner products when the group G over $\mathbf{Q}_p$ is the product of a torus over $\mathbf{Q}_p$ with some $\mathbf{Q}_p$-simple groups, since every invariant inner product on G is the orthogonal direct sum of invariant inner products on the torus and on the $\mathbf{Q}_p$-simple factors by Lemma 7.

Now let G be any connected reductive group over $\mathbf{Q}_p$; we will prove the theorem for any (λ_1, b) in G and any invariant inner product on G. As mentioned in section 7, G is the quotient of a product of a torus and some $\mathbf{Q}_p$-simple groups by a finite central subgroup; by dividing out a little more, we see (what is more useful here) that the quotient G' of G by some finite central subgroup is the product of a torus and some $\mathbf{Q}_p$-simple groups. Then (λ_1, b) maps to a pair (λ'_1, b') in G', the invariant inner product on G determines an invariant inner product on G', and we know that (λ'_1, b') is weakly admissible if and only if the second condition in the theorem for G' is satisfied. Now the second condition is satisfied for G if and only if it is satisfied for G', because the corresponding map $J \to J'$ is an isogeny, and so every one-parameter subgroup of J' has a positive multiple which lifts to J. So the theorem will follow if we can show that (λ_1, b) is weakly admissible if and only if (λ'_1, b') is.

There are more direct ways to see this, but one way which we have at hand now is that we have already proved the theorem for at least one inner product on G, namely one coming from a faithful representation of G. Thus (λ_1, b) is weakly admissible

$\iff$ the second condition of the theorem is satisfied for G and the inner product just mentioned

$\iff$ the second condition of the theorem is satisfied for G' and the corresponding inner product on G'

$\iff$ (λ'_1, b') is weakly admissible (since we have proved the theorem for arbitrary invariant inner products on G'). QED.

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Department of Mathematics, University of Chicago, 5734 S. University Ave., Chicago, IL 60637

totaro@math.uchicago.edu