Cohomology of semidirect product groups

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We give examples of semidirect product groups $G \ltimes A$ such that the Hochschild-Serre spectral sequence $H^*(G, H^*A) \Rightarrow H^*(G \ltimes A)$ for $\mathbf{Z}/p$-cohomology has nonzero differentials. Until now, few such examples have been known, especially when the normal subgroup A is abelian. In particular, Benson and Feshbach [2] mentioned that in all known semidirect products with A abelian, the spectral sequence satisfies:

(1) All differentials after d_2 are 0.

(2) All differentials are 0 if $A = (S^1)^n$. (To be consistent with the notation for discrete groups A, H^*A here means the cohomology of the classifying space of A.)

(3) All differentials are 0 if $A = (\mathbf{Z}/2)^n$ and we consider cohomology with $\mathbf{Z}/2$ coefficients.

We give examples to show that all three statements can fail. In fact, there can be nonzero differentials at d_p or later in all of these cases. I expect that there can be nonzero differentials arbitrarily far along in the spectral sequence in all of these cases, but the problem remains open. (For semidirect products $G \ltimes A$ with A not abelian, Benson and Feshbach [2] gave examples of nonzero differentials arbitrarily far along in the spectral sequence for $\mathbf{Z}/2$-cohomology.)

It turns out that there is a very general reason why there will be nonzero differentials in some examples. If X is a G-space, then $H^*(G, C^*(X))$ admits Steenrod operations compatible with those on H^*G because it is the cohomology of the space $(X \times EG)/G$, whereas there is no reason for $H^*(G, M)$ to have Steenrod operations for a general G-module M. Thus Steenrod operations provide a fundamental obstruction for a G-module to be the representation of G on the cohomology of a G-space, as G. Carlsson found [4]; there is a useful exposition by Benson and Habegger [3]. If a semidirect product $G \ltimes A$ has the G-action on A given by the dual of such a G-module, we can show that there must be nonzero differentials in the Hochschild-Serre spectral sequence.

It is interesting to contrast these examples with Nakaoka's theorem that the Hochschild-Serre spectral sequence has no differentials for any wreath product $G \ltimes H^n$ ([6], p. 50). Here G and H are any finite groups and G acts on H^n through a permutation representation $G \hookrightarrow S_n$. It would be good to characterize algebraically the class of G-modules M over $\mathbf{Z}/p$, say for a p-group G, such that the semidirect product $G \ltimes M$ has no differentials in the spectral sequence: it seems to be fairly close to the class of permutation modules, but there are some other interesting examples.

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1 Lemma on Steenrod operations

Let p be a prime number. Throughout this paper we write H^*X for $H^*(X, \mathbf{Z}/p)$. Also, let $H^{\cdot}X$ be $H^*(X, \mathbf{Z}/2)$ if $p = 2$, and let it be the even-dimensional subring $H^{ev}(X, \mathbf{Z}/p)$ is p is odd. Thus $H^{\cdot}X$ is always a commutative ring. Finally, for p odd, $P^i : H^n X \to H^{n+2(p-1)i} X$ are the usual Steenrod operations, and for $p = 2$ we use the same notation P^i to mean $P^i = Sq^i : H^n X \to H^{n+i} X$.

We need the following lemma, which is a variant of Proposition 3 in Landweber and Stong [7]. The proof is short enough to repeat here. Recall that the radical of an ideal I in a commutative ring R is the set of $a \in R$ such that $a^n \in I$ for some $n \geq 1$.

Lemma 1 *Let p be a prime number, and let H^*X denote $H^*(X, \mathbf{Z}/p)$. Let $X \to Y$ be a map of spaces. Let $M \subset H^*X$ be any finitely generated graded $H^{\cdot}Y$-submodule. Then the radical of the annihilator of M is an ideal in $H^{\cdot}Y$ which is closed under the Steenrod operations P^i, $i \geq 0$.*

The interesting thing is that M is not assumed to be closed under the Steenrod operations, and as a result the annihilator of M is generally not closed under the Steenrod operations; but the radical of the annihilator of M behaves better.

Proof. It suffices to prove that the radical of the annihilator in $H^{\cdot}Y$ of a single element $x \in H^*X$ is closed under all P^i's. For $2i > \dim x$ (or $i > \dim x$, in case $p = 2$) we have $P^i x = 0$. So there is a positive integer r large enough that $P^i x = 0$ for $i \geq p^r$. Then, for any $a \in H^{\cdot}Y$ and $i \geq 0$, we have

$$P^{ip^r}(a^{p^r} x) = (P^i a)^{p^r} x.$$

This follows from the Cartan identity for Steenrod operations, which says that the total Steenrod operation $P = 1 + P^1 + P^2 + \cdots$ is a ring homomorphism from H^*X to itself, so that

$$\begin{aligned} P(a^{p^r} x) &= P(a)^{p^r} P(x) \\ &= (\sum_{k \geq 0} (P^k a)^{p^r})(\sum_{0 \leq l < p^r} P^l x). \end{aligned}$$

The earlier identity follows by equating terms in the appropriate dimension. Now if a belongs to the radical of the annihilator of x, we may assume that $a^{p^r} x = 0$. The identity then shows that $P^i a$ is in the radical of the annihilator of x for all $i \geq 0$. QED

2 Cohomology of semidirect product groups

Let $G \ltimes A$ be a semidirect product of groups, where we need not assume that A is abelian, although that is where I have applications for the theorem. Let $H^*G = H^*(G, \mathbf{Z}/p)$ for a fixed prime number p, and let $H^{\cdot}G$ be H^*G for $p = 2$, $H^{ev}G$ for p odd.

A group G is defined to be of type VFP for $\mathbf{Z}/p$-coefficients if it has a subgroup of finite index whose total cohomology with coefficients in any module of finite dimension over $\mathbf{Z}/p$ is finite-dimensional. Finite groups as well as arithmetic groups, such as $GL_n\mathbf{Z}$, are examples of groups of type VFP.

Theorem 1 *Suppose that G has type VFP for $\mathbf{Z}/p$ and that H^iA is finite-dimensional for each i. Let r be the smallest number ≥ 1 such that $H^rA \neq 0$. If the Hochschild-Serre spectral sequence for computing $H^*(G \ltimes A)$ has all differentials into $H^*(G, H^rA)$ equal to 0, then the radical of the annihilator of the $H^{\cdot}G$-module $H^*(G, H^rA)$ is closed under the Steenrod operations P^i.*

Proof. Venkov and Evens proved that for finite groups G, $H^{\cdot}G$ is a noetherian ring and $H^*(G, M)$ is a finitely generated $H^{\cdot}G$-module for all $\mathbf{Z}/pG$-modules M of finite dimension over $\mathbf{Z}/p$ ([1], p. 130). The Hochschild-Serre spectral sequence shows that these properties generalize to groups G of type VFP for $\mathbf{Z}/p$.

Since we have a semidirect product, the 0th row of the spectral sequence, H^*G, splits off from $H^*(G \ltimes A)$ as an H^*G-module in a natural way. The remaining piece of $H^*(G \ltimes A)$ has a filtration by H^*G-submodules, with the bottom piece of the filtration isomorphic to $H^*(G, H^rA)/(\text{all differentials})$. If, as we assume, there are no differentials mapping into the rth row, then we have exhibited $H^*(G, H^rA)$ as an H^*G-submodule of $H^*(G \ltimes A)$.

By Lemma 1, even though the Steenrod operations need not map $H^*(G, H^rA)$ into itself, the radical of the annihilator of $H^*(G, H^rA)$ in $H^{\cdot}G$ is closed under the Steenrod operations. QED

Corollary 1 *For each prime number p, there are semidirect products $(\mathbf{Z}/p)^2 \ltimes (\mathbf{Z}/p)^n$, $(\mathbf{Z}/p)^2 \ltimes \mathbf{Z}^n$, and $(\mathbf{Z}/p)^2 \ltimes (S^1)^n$ such that the Hochschild-Serre spectral sequence with $\mathbf{Z}/p$ coefficients does not degenerate. More precisely there will be nonzero differentials mapping into $H^*(G, H^1A)$ in the first two cases and into $H^*(G, H^2A)$ in the last case. We can take $n = 2p^2$.*

Proof. Let $G = (\mathbf{Z}/p)^2$. Following Benson [1], pp. 190-195, we will exhibit a $\mathbf{Z}G$-module L_ζ which is free as a $\mathbf{Z}$-module such that the radical of the annihilator of the $H^{\cdot}G$-module $H^*(G, L_\zeta \otimes \mathbf{Z}/p)$ is not closed under the Steenrod operations P^i. (One can define a module L_ζ with this property for any finite group G of p-rank ≥ 2, but we will just prove what we need for $G = (\mathbf{Z}/p)^2$.) Then, if we define an abelian group A with G-action by $A = Hom(L_\zeta, \mathbf{Z}/p)$, $A = Hom(L_\zeta, \mathbf{Z})$, or $A = BHom(L_\zeta, \mathbf{Z})$ (in the last case $A \cong (S^1)^n$), then the lowest-dimensional cohomology of A (H^rA where $r = 1, 1, 2$, respectively) is isomorphic to $L_\zeta \otimes \mathbf{Z}/p$ as a G-module. By Theorem 1, there are nonzero differentials in the spectral sequence of the extension $G \ltimes A$ with $\mathbf{Z}/p$ coefficients in these three cases. In fact there are nonzero differentials mapping into $H^*(G, H^rA)$.

We define the $\mathbf{Z}G$-module L_ζ as follows. Let $x, y \in H^2G$ span the space of Bocksteins of elements of H^1, so that x and y generate a polynomial subring of H^*G, and let ζ be a homogenous irreducible polynomial in x, y over $\mathbf{Z}/p$ of degree $d > 1$. Then ζ gives an element of $H^{2d}G$, which even lifts to $H^{2d}(G, \mathbf{Z})$ since x and y are integral classes. Fix such a lift. Let

$$\cdots \to P_1 \to P_0 \to \mathbf{Z} \to 0$$

be a projective resolution of $\mathbf{Z}$ as a $\mathbf{Z}G$-module, and let $\Omega^i\mathbf{Z}$ be the image of P_i in P_{i-1}; it may depend on the resolution, although that is irrelevant to us. Then the lift of ζ in $H^{2d}(G, \mathbf{Z})$ can be represented by a map $\Omega^{2d}\mathbf{Z} \rightarrow \mathbf{Z}$ of $\mathbf{Z}G$-modules. Let L_ζ be the kernel, so that we have a short exact sequence of $\mathbf{Z}G$-modules,

$$0 \rightarrow L_\zeta \rightarrow \Omega^{2d}\mathbf{Z} \rightarrow \mathbf{Z} \rightarrow 0.$$

These are torsion-free abelian groups, because $\Omega^{2d}\mathbf{Z}$ is a submodule of P_{2d-1}. The cohomology of G with coefficients in $(\Omega^{2d}\mathbf{Z}) \otimes \mathbf{Z}/p$ is just H^*G shifted up by $2d$, at least in dimensions $\geq 2d+1$, and the map

$$\zeta : (\Omega^{2d}\mathbf{Z}) \otimes \mathbf{Z}/p \rightarrow \mathbf{Z}/p$$

gives a map

$$H^iG = H^{i+2d}(G, (\Omega^{2d}\mathbf{Z}) \otimes \mathbf{Z}/p) \rightarrow H^{i+2d}(G, \mathbf{Z}/p)$$

which is multiplication by $\zeta \in H^{2d}G$. Multiplication by ζ is an injective map on H^*G (for $p = 2$, H^*G is a polynomial ring; for p odd, H^*G is the tensor product of a polynomial ring and an exterior algebra, with ζ in the polynomial subring). So the short exact sequence above, which remains exact on tensoring with $\mathbf{Z}/p$, determines $H^*(G, L_\zeta \otimes \mathbf{Z}/p)$ in high dimensions: for $i \geq 2d + 2$,

$$H^i(G, L_\zeta \otimes \mathbf{Z}/p) = H^{i-1}G/(\zeta).$$

Knowing $H^*(G, L_\zeta \otimes \mathbf{Z}/p)$ in dimensions $\geq 2d + 2$ is enough if we only want to know the radical of the annihilator of the $H^{\cdot}G$-module $H^*(G, L_\zeta \otimes \mathbf{Z}/p)$; namely, this radical is the ideal $(\sqrt{\zeta})$ in $H^{\cdot}G = H^*G$ for $p = 2$, or (ζ) in $H^{\cdot}G = H^{ev}G$ for p odd. But Serre [11] showed that if an ideal in $H^{\cdot}G$ is closed under the Steenrod operations, then the corresponding algebraic subset of Spec $H^{\cdot}G = A^2_{\mathbf{Z}/p}$ is a finite union of $\mathbf{Z}/p$-linear subspaces. Since the polynomial ζ is irreducible of degree > 1 over $\mathbf{Z}/p$, Serre's theorem shows that the radical of the annihilator of the $H^{\cdot}G$-module $H^*(G, L_\zeta \otimes \mathbf{Z}/p)$ is not closed under the Steenrod operations, which is the property of L_ζ we want.

Specifically, let ζ be an irreducible quadratic polynomial over $\mathbf{Z}/p$, so that $d = 2$ above. There is a resolution of $\mathbf{Z}$ over $\mathbf{Z}G$, where $G = (\mathbf{Z}/p)^2$, of the form

$$\cdots \rightarrow (\mathbf{Z}G)^3 \rightarrow (\mathbf{Z}G)^2 \rightarrow (\mathbf{Z}G)^1 \rightarrow \mathbf{Z} \rightarrow 0,$$

and one computes that $\Omega^4\mathbf{Z}$ is a $\mathbf{Z}G$-module of $\mathbf{Z}$-rank $2p^2 + 1$ for this resolution. So L_ζ is a $\mathbf{Z}G$-module of $\mathbf{Z}$-rank $2p^2$, and we can take $A = (\mathbf{Z}/p)^{2p^2}$, $A = \mathbf{Z}^{2p^2}$, or $A = (S^1)^{2p^2}$ for our example. QED

Steve Siegel pointed out to me that in the special case of semidirect products $(\mathbf{Z}/2)^2 \ltimes (\mathbf{Z}/2)^n$, we can take the cohomology classes x and y in the above construction to be in H^1 rather than H^2, with the result that there is a semidirect product of this type with nonzero Hochschild-Serre differentials for $n = 4$, rather than $n = 2p^2 = 8$.

3 Comments

At this point we have answered negatively two of the three questions raised in [2]: the spectral sequence of a semidirect product $G \ltimes A$ does not always degenerate for $A = (S^1)^n$, nor for $A = (\mathbf{Z}/2)^n$ with $\mathbf{Z}/2$ coefficients. (Examples where the differential d_2 was nonzero were known before for $A = \mathbf{Z}^n$ ([5] and [9], pp. 28-29) and for $A = (\mathbf{Z}/p)^n$, at least when $p \geq 5$ [10].)

We can also answer no to the remaining question asked in [2], whether the d_2 differential is the only one which can be nonzero in the spectral sequence for semidirect products by an abelian group. The point is that for cohomology with $\mathbf{Z}/p$ coefficients, the only differentials which can be nonzero in the spectral sequence for a semidirect product $G \ltimes \mathbf{Z}^n$ are the d_i's with $i \equiv 1 \pmod{p-1}$, starting with d_p. This is an easy consequence of Lieberman's trick, that is, of the action of the multiplicative monoid of the positive integers on $G \ltimes \mathbf{Z}^n$ by fixing G and acting in the obvious way on $\mathbf{Z}^n$ ([8], p. 262). Thus, for the semidirect products $(\mathbf{Z}/p)^2 \ltimes \mathbf{Z}^n$ produced in Corollary 1, there is a nonzero differential at d_p or later. The same argument shows that for $(\mathbf{Z}/p)^2 \ltimes (S^1)^n$ as in Corollary 1, there is a nonzero differential at d_{2p-1} or later. (In this case the possibly nonzero differentials are d_i for $i \equiv 1 \pmod{2(p-1)}$.)

Also, for semidirect products $G \ltimes (\mathbf{Z}/p)^n$ as constructed in Corollary 1, there will be a nonzero differential at d_p or later. The point is that, in the Corollary, the G-action on $(\mathbf{Z}/p)^n$ lifts to an action on $A = \mathbf{Z}^n$. The resulting homomorphism $G \ltimes A \to G \ltimes (A/p)$ gives a map of spectral sequences which is an isomorphism on row 1 of the E_2 term:

$$H^i(G, H^1(A/p)) \overset{\cong}{\to} H^i(G, H^1 A).$$

Since there are no differentials into $H^i(G, H^1 A)$ until d_p or later, there are no differentials into $H^i(G, H^1(A/p))$ until d_p or later. Moreover Corollary 1 says that there will be a nonzero differential into $H^i(G, H^1(A/p))$ sometime, thus necessarily at d_p or later.

The question remains whether the $\mathbf{Z}/p$-cohomology spectral sequence for semidirect products $G \ltimes (\mathbf{Z}/p)^n$ or $G \ltimes \mathbf{Z}^n$ can have nonzero differentials after d_p. I expect that there can be nonzero differentials arbitrarily far along in the spectral sequence.

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