Essential dimension of the spin groups in characteristic 2

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The essential dimension of an algebraic group G is a measure of the number of parameters needed to describe all G-torsors over all fields. A major achievement of the subject was the calculation of the essential dimension of the spin groups over a field of characteristic not 2, started by Brosnan, Reichstein, and Vistoli, and completed by Chernousov, Merkurjev, Garibaldi, and Guralnick [3, 4, 7], [18, Theorem 9.1].

In this paper, we determine the essential dimension of the spin group $\mathrm{Spin}(n)$ for $n \geq 15$ over an arbitrary field (Theorem 2.1). We find that the answer is the same in all characteristics. In contrast, for the groups $O(n)$ and $SO(n)$, the essential dimension is smaller in characteristic 2, by Babic and Chernousov [1].

In characteristic not 2, the computation of essential dimension can be phrased to use a natural finite subgroup of $\mathrm{Spin}(2r+1)$, namely an extraspecial 2-group, a central extension of $(\mathbf{Z}/2)^{2r}$ by $\mathbf{Z}/2$. A distinctive feature of the argument in characteristic 2 is that the analogous subgroup is a finite group scheme, a central extension of $(\mathbf{Z}/2)^r \times (\mu_2)^r$ by μ_2, where μ_2 is the group scheme of square roots of unity.

In characteristic not 2, Rost and Garibaldi computed the essential dimension of $\mathrm{Spin}(n)$ for $n \leq 14$ [6, Table 23B], where case-by-case arguments seem to be needed. We show in Theorem 4.1 that for $n \leq 10$, the essential dimension of $\mathrm{Spin}(n)$ is the same in characteristic 2 as in characteristic not 2. It would be interesting to compute the essential dimension of $\mathrm{Spin}(n)$ in the remaining cases, $11 \leq n \leq 14$ in characteristic 2.

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1 Essential dimension

Let G be an affine group scheme of finite type over a field k. Write $H^1(k, G)$ for the set of isomorphism classes of G-torsors over k in the fppf topology. For G smooth over k, this is also the set of isomorphism classes of G-torsors over k in the etale topology.

Following Reichstein, the *essential dimension* $\mathrm{ed}(G)$ is the smallest natural number r such that for every G-torsor ξ over an extension field E of k, there is a subfield $k \subset F \subset E$ such that ξ is isomorphic to some G-torsor over F extended to E, and F has transcendence degree at most r over k. (It is essential that E is allowed to be any extension field of k, not just an algebraic extension field.) There are several survey articles on essential dimension, including [19, 17].

For example, let q_0 be a quadratic form of dimension n over a field k of characteristic not 2. Then $O(q_0)$-torsors can be identified with quadratic forms of dimension n, up to isomorphism. (For convenience, we sometimes write $O(n)$ for $O(q_0)$.) Thus the essential dimension of $O(n)$ measures the number of parameters needed to describe all quadratic forms of dimension n. Indeed, every quadratic form of dimension n over a field of characteristic not 2 is isomorphic to a diagonal form $\langle a_1, \ldots, a_n \rangle$. It follows that the orthogonal group $O(n)$ in characteristic not 2 has essential dimension at most n; in fact, $O(n)$ has essential dimension equal to n, by one of the first computations of essential dimension [19, Example 2.5]. Reichstein also showed that the connected group $SO(n)$ in characteristic not 2 has essential dimension $n-1$ for $n \geq 3$ [19, Corollary 3.6].

For another example, for an integer $n \geq 2$ and any field k, the group scheme μ_n of nth roots of unity is smooth over k if and only if n is invertible in k. Independent of that, $H^1(k, \mu_n)$ is always isomorphic to $k^*/(k^*)^n$. From that description, it is immediate that μ_n has essential dimension at most 1 over k. It is not hard to check that the essential dimension is in fact equal to 1.

One simple bound is that for any generically free representation V of a group scheme G over k (meaning that G acts freely on a nonempty open subset of V), the essential dimension of G is at most $\dim(V) - \dim(G)$ [18, Proposition 5.1]. It follows, for example, that the essential dimension of any affine group scheme of finite type over k is finite.

For a prime number p, the p-essential dimension $\mathrm{ed}_p(G)$ is a simplified invariant, defined by "ignoring field extensions of degree prime to p". In more detail, for a G-torsor ξ over an extension field E of k, define the p-essential dimension $\mathrm{ed}_p(\xi)$ to be the smallest number r such that there is a finite extension E'/E of degree prime to p such that ξ over E' comes from a G-torsor over a subfield $k \subset F \subset E'$ of transcendence degree at most r over k. Then the *p-essential dimension* $\mathrm{ed}_p(G)$ is defined to be the supremum of the p-essential dimensions of all G-torsors over all extension fields of k.

The *spin group* $\mathrm{Spin}(n)$ is the simply connected double cover of $SO(n)$. It was a surprise when Brosnan, Reichstein, and Vistoli showed that the essential dimension of $\mathrm{Spin}(n)$ over a field k of characteristic not 2 is exponentially large, asymptotic to $2^{n/2}$ as n goes to infinity [3]. As an application, they showed that the number of "parameters" needed to describe all quadratic forms of dimension $2r$ in I^3 over all fields is asymptotic to 2^r.

We now turn to quadratic forms over a field which may have characteristic 2. Define a quadratic form (q, V) over a field k to be *nondegenerate* if the radical $V^\perp$ of the associated bilinear form is 0, and *nonsingular* if $V^\perp$ has dimension at most 1 and q is nonzero on any nonzero element of $V^\perp$. (In characteristic not 2, nonsingular and nondegenerate are the same.) The orthogonal group is defined as the automorphism group scheme of a nonsingular quadratic form [13, section VI.23]. For example, over a field k of characteristic 2, the quadratic form

$$x_1x_2 + x_3x_4 + \cdots + x_{2r-1}x_{2r}$$

is nonsingular of even dimension $2r$, while the form

$$x_1x_2 + x_3x_4 + \cdots + x_{2r-1}x_{2r} + x_{2r+1}^2$$

is nonsingular of odd dimension $2r+1$, with $V^\perp$ of dimension 1. The *split* orthogonal group over k is the automorphism group of one of these particular quadratic forms.

Babic and Chernousov computed the essential dimension of $O(n)$ and the smooth connected subgroup $O^+(n)$ over an infinite field k of characteristic 2 [1]. (We also write $SO(n)$ for $O^+(n)$ by analogy with the case of characteristic not 2, even though the whole group $O(2r)$ is contained in $SL(2r)$ in characteristic 2.) The answer is smaller than in characteristic not 2. Namely, $O(2r)$ has essential dimension $r+1$ (not $2r$) over k. Also, $O^+(2r)$ has essential dimension $r+1$ for r even, and either r or $r+1$ for r odd, not $2r-1$. Finally, the group scheme $O(2r+1)$ has essential dimension $r+2$ over k, and $O^+(2r+1)$ has essential dimension $r+1$. The lower bounds here are difficult, while the upper bounds are straightforward. For example, to show that $O(2r)$ has essential dimension at most $r+1$ in characteristic 2, write any quadratic form of dimension $2r$ as a direct sum of 2-dimensional forms, thus reducing the structure group to $(\mathbf{Z}/2)^r \times (\mu_2)^r$, and then use that the group $(\mathbf{Z}/2)^r$ has essential dimension only 1 over an infinite field of characteristic 2 [1, proof of Proposition 13.1].

In this paper, we determine the essential dimension of $\text{Spin}(n)$ in characteristic 2 for $n \leq 10$ or $n \geq 15$. Surprisingly, in view of what happens for $O(n)$ and $O^+(n)$, the results for spin groups are the same in characteristic 2 as in characteristic not 2. For $n \leq 10$, the lower bound for the essential dimension is proved by constructing suitable cohomological invariants. It is not known whether a similar approach is possible for $n \geq 15$, either in characteristic 2 or in characteristic not 2.

2 Main result

Theorem 2.1. *Let k be a field. For every integer n at least 15, the essential dimension of the split group* $\text{Spin}(n)$ *over k is given by:*

$$\text{ed}_2(\text{Spin}(n)) = \text{ed}(\text{Spin}(n)) = \begin{cases} 2^{n-1} - n(n-1)/2 & \text{if } n \text{ is odd;} \\ 2^{(n-2)/2} - n(n-1)/2 & \text{if } n \equiv 2 \pmod 4; \\ 2^{(n-2)/2} + 2^m - n(n-1)/2 & \text{if } n \equiv 0 \pmod 4, \end{cases}$$

where 2^m is the largest power of 2 dividing n.

Proof. For k of characteristic 0, this was proved by Chernousov and Merkurjev, sharpening the results of Brosnan, Reichstein, and Vistoli [4, Theorem 2.2]. Their argument works in any characteristic not 2, using the results of Garibaldi and Guralnick for the upper bounds [7]. Namely, Garibaldi and Guralnick showed that for any field k and any n at least 15, $\text{Spin}(n)$ acts generically freely on the spin representation for n odd, on each of the two half-spin representations if $n \equiv 2 \pmod 4$, and on the direct sum of a half-spin representation and the standard representation if $n \equiv 0 \pmod 4$. Moreover, for n at least 20 with $n \equiv 0 \pmod 4$, $\text{HSpin}(n) = \text{Spin}(n)/\mu_2$ (the quotient different from $O^+(n)$) acts generically freely on a half-spin representation [7, Theorem 1.1].

It remains to consider a field k of characteristic 2. Garibaldi and Guralnick's result gives the desired upper bound in most cases. Namely, for n odd and at least 15, the spin representation has dimension $2^{(n-1)/2}$, and so $\text{ed}(\text{Spin}(n)) \leq 2^{(n-1)/2} - \dim(\text{Spin}(n)) = 2^{(n-1)/2} - n(n-1)/2$. For $n \equiv 2 \pmod 4$, the half-spin representations have dimension $2^{(n-2)/2}$, and so $\mathrm{ed}(\mathrm{Spin}(n)) \leq 2^{(n-2)/2} - n(n-1)/2$. For $n = 16$, since the spin group acts generically freely on the direct sum of a half-spin representation and the standard representation, $\mathrm{ed}(\mathrm{Spin}(n)) \leq 2^{(n-2)/2} + n - n(n-1)/2$ $(= 24)$.

For n at least 20 and divisible by 4, the optimal upper bound requires more effort. The following argument is modeled on Chernousov and Merkurjev's characteristic zero argument [4, Theorem 2.2]. Namely, consider the map of exact sequences of k-group schemes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \mathrm{Spin}(n) & \longrightarrow & \mathrm{HSpin}(n) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle =} & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & O^+(n) & \longrightarrow & \mathrm{PGO}^+(n) & \longrightarrow & 1. \end{array}$$

Since $\mathrm{HSpin}(n)$ acts generically freely on a half-spin representation, which has dimension $2^{(n-2)/2}$, we have $\mathrm{ed}(\mathrm{HSpin}(n)) \leq 2^{(n-2)/2} - n(n-1)/2$.

By Chernousov-Merkurjev or independently Lötscher, for any normal subgroup scheme C of an affine group scheme G over a field k,

$$\mathrm{ed}(G) \leq \mathrm{ed}(G/C) + \max\, \mathrm{ed}\,[E/G],$$

where the maximum runs over all field extensions F of k and all G/C-torsors E over F [4, Proposition 2.1], [15, Example 3.4]. Thus $[E/G]$ is a gerbe over F banded by C.

Identifying $H^2(K, \mu_p)$ with the p-torsion in the Brauer group of K, we can talk about the index of an element of $H^2(K, \mu_p)$, meaning the degree of the corresponding division algebra over K. For a prime number p and a nonzero element E of $H^2(K, \mu_p)$ over a field K, the essential dimension (or also the p-essential dimension) of the corresponding μ_p-gerbe over K is equal to the index of E, by Karpenko and Merkurjev [11, Theorems 2.1 and 3.1].

By the diagram above, for any field F over k, the image of the connecting map

$$H^1(F, \mathrm{HSpin}(n)) \to H^2(F, \mu_2) \subset \mathrm{Br}(F)$$

is contained in the image of the other connecting map

$$H^1(F, PGO^+(n)) \to H^2(F, \mu_2) \subset \mathrm{Br}(F).$$

In the terminology of the Book of Involutions, the image of the latter map consists of the classes $[A]$ of all central simple F-algebras A of degree n with a quadratic pair (σ, f) of trivial discriminant [13, section 29.F]. Any torsor for $PGO^+(n)$ is split by a field extension of degree a power of 2, by reducing to the corresponding fact about quadratic forms. So $\mathrm{ind}(A)$ must be a power of 2, but it also divides n, and so $\mathrm{ind}(A) \leq 2^m$, where 2^m is the largest power of 2 dividing n. We conclude that

$$\begin{aligned} \mathrm{ed}(\mathrm{Spin}(n)) &\leq \mathrm{ed}(\mathrm{HSpin}(n)) + 2^m \\ &\leq 2^{(n-2)/2} - n(n-1)/2 + 2^m. \end{aligned}$$

This completes the proof of the upper bound in Theorem 2.1.

We now prove the corresponding lower bound for the 2-essential dimension of the spin group over a field k of characteristic 2. Since $\mathrm{ed}_2(\mathrm{Spin}(n)) \leq \mathrm{ed}(\mathrm{Spin}(n))$, this will imply that the 2-essential dimension and the essential dimension are both equal to the number given in Theorem 2.1.

Write $O(2r)$ for the orthogonal group of the quadratic form $x_1x_2+x_3x_4+\cdots+x_{2r-1}x_{2r}$ over k, and $O(2r+1)$ for the orthogonal group of $x_1x_2+x_3x_4+\cdots+x_{2r-1}x_{2r}+x_{2r+1}^2$. Then we have an inclusion $O(2r)\subset O(2r+1)$. Note that $O(2r)$ is smooth over k, with $O(2r)/O^+(2r)\cong \mathbf{Z}/2$. The group scheme $O(2r+1)$ is not smooth over k, but it contains a smooth connected subgroup $O^+(2r+1)$ with $O(2r+1)\cong O^+(2r+1)\times \mu_2$. It follows that $O(2r)$ is contained in $O^+(2r+1)$. Using the subgroup $\mathbf{Z}/2\times\mu_2$ of $O(2)$, we have a k-subgroup scheme $K:=(\mathbf{Z}/2\times\mu_2)^r\subset O(2r)\subset O^+(2r+1)$. Let G be the inverse image of K in the double cover $\mathrm{Spin}(2r+1)$ of $O^+(2r+1)$. Thus G is a central extension

$$1\to\mu_2\to G\to(\mathbf{Z}/2)^r\times(\mu_2)^r\to 1.$$

(Essentially the same "finite Heisenberg group scheme" appeared in the work of Mumford and Sekiguchi on abelian varieties [20, Appendix A].)

To describe the structure of G in more detail, think of $K=(\mu_2)^r$ as the 2-torsion subgroup scheme of a fixed maximal torus $T_{SO}\cong (G_m)^r$ in $O^+(2r+1)$, where G_m is the multiplicative group. The chararacter group of T_{SO} is the free abelian group $\mathbf{Z}\{x_1,\ldots,x_r\}$, and the Weyl group $W=N(T_{SO})/T_{SO}$ of $O^+(2r+1)$ is the semidirect product $S_r\ltimes(\mathbf{Z}/2)^r$. Here S_r permutes the characters $x_1,\ldots,x_r$ of T_{SO}, and the subgroup $E_r=(\mathbf{Z}/2)^r$ of W, with generators $\epsilon_1,\ldots,\epsilon_r$, acts by: ϵ_i changes the sign of x_i and fixes x_j for $j\neq i$. The character group of $K=T_{SO}[2]$ is $\mathbf{Z}/2\{x_1,\ldots,x_r\}$. The group E_r centralizes K, and the group $(\mathbf{Z}/2)^r\times(\mu_2)^r\subset O^+(2r+1)$ above is $E_r\times K$.

Let L be the inverse image of K in $\mathrm{Spin}(2r+1)$, which is contained in a maximal torus T of $\mathrm{Spin}(2r+1)$, the inverse image of T_{SO}. The character group $X^*(T)$ is

$$\mathbf{Z}\{x_1,\ldots,x_r,A\}/(2A=x_1+\cdots+x_r).$$

Therefore, the character group $X^*(L)$ is

$$\mathbf{Z}\{x_1,\ldots,x_r,A\}/(2x_i=0,\ 2A=x_1+\cdots+x_r).$$

(Thus $X^*(L)$ is isomorphic to $(\mathbf{Z}/4)\times(\mathbf{Z}/2)^{r-1}$, and so L is isomorphic to $\mu_4\times(\mu_2)^{r-1}$.) The Weyl group W of $\mathrm{Spin}(2r+1)$ is the same as that of $O^+(2r+1)$, namely $S_r\ltimes E_r$. In particular, the element ϵ_i of E_r acts on $X^*(T)$ by changing the sign of x_i and fixing x_j for $j\neq i$, and hence it sends A to $A-x_i$.

The subset S of $X^*(L)$ corresponding to characters of L which are faithful on the center μ_2 of L is the complement of the subgroup $X^*(K)=\mathbf{Z}/2\{x_1,\ldots,x_r\}$. Therefore, S has order 2^r. The group $E_r=(\mathbf{Z}/2)^r$ acts freely and transitively on S, since

$$\left(\prod_{i\in I}\epsilon_i\right)(A)=A-\sum_{i\in I}x_i$$

for any subset I of $\{1,\ldots,r\}$.

The group $G=E_r\cdot L$ is the central extension considered above. Now, let V be a representation of G over k on which the center $\mu_2\subset L$ acts faithfully by scalars. Then the restriction of V to L is fixed (up to isomorphism) by the action of E_r on $X^*(L)$. By the previous paragraph, the 2^r 1-dimensional representations of L that are nontrivial on the center μ_2 all occur with the same multiplicity in V. Therefore, V has dimension a multiple of 2^r. This bound is optimal, since the spin representation W of $\mathrm{Spin}(2r+1)$ has dimension 2^r over k, and the center μ_2 acts faithfully by scalars on W.

We use the following result of Merkurjev's [16, Theorem 5.2], [11, Remark 4.5]. (The first reference covers the case of the group scheme μ_p in characteristic p, as needed here.)

Theorem 2.2. *Let k be a field and p be a prime number. Let $1 \to \mu_p \to G \to Q \to 1$ be a central extension of affine group schemes over k. For a field extension K of k, let $\partial_K \colon H^1(K,Q) \to H^2(K,\mu_p)$ be the boundary homomorphism in fppf cohomology. Then the maximal value of the index of $\partial_K(E)$, as K ranges over all field extensions of k and E ranges over all Q-torsors over K, is equal to the greatest common divisor of the dimensions of all representations of G on which μ_p acts by its standard representation.*

As mentioned above, for a prime number p and a nonzero element E of $H^2(K,\mu_p)$ over a field K, the essential dimension (or also the p-essential dimension) of the corresponding μ_p-gerbe over K is equal to the index of E.

Finally, consider a central extension $1 \to \mu_p \to G \to Q \to 1$ of finite group schemes over a field k. Generalizing an argument of Brosnan-Reichstein-Vistoli, Karpenko and Merkurjev showed that the p-essential dimension of G (and hence the essential dimension of G) is at least the p-essential dimension of the μ_p-gerbe over K associated to any Q-torsor over any field K/k [11, Theorem 4.2]. By the analysis above of representations of the finite subgroup scheme G of $\mathrm{Spin}(2r+1)$ over a field k of characteristic 2, we find that $\mathrm{ed}_2(G) \geq 2^r$. For a closed subgroup scheme G of a group scheme L over a field k and any prime number p, we have $\mathrm{ed}_p(L)+\dim(L) \geq \mathrm{ed}_p(G)+\dim(G)$ [17, Corollary 4.3] (which covers the case of fppf torsors for non-smooth group schemes, as needed here). Applying this to the subgroup scheme G of $\mathrm{Spin}(2r)$, we conclude that $\mathrm{ed}_2(\mathrm{Spin}(2r+1)) \geq 2^r - \dim(\mathrm{Spin}(2r+1)) = 2^r - r(2r+1)$. Combining this with the upper bound discussed above, we have

$$\mathrm{ed}(\mathrm{Spin}(2r+1)) = \mathrm{ed}_2(\mathrm{Spin}(2r+1)) = 2^r - r(2r+1)$$

for $r \geq 7$.

The proof of the lower bound for $\mathrm{ed}_2(\mathrm{Spin}(2r))$ when r is odd is similar. The intersection of the subgroup $K = (\mu_2 \times \mathbf{Z}/2)^r \subset O(2r)$ with $O^+(2r)$ is $K_1 \cong (\mu_2)^r \times (\mathbf{Z}/2)^{r-1}$, where $(\mathbf{Z}/2)^{r-1}$ denotes the kernel of the sum $(\mathbf{Z}/2)^r \to \mathbf{Z}/2$. As a result, the double cover $\mathrm{Spin}(2r)$ contains a subgroup G_1 which is a central extension

$$1 \to \mu_2 \to G_1 \to (\mathbf{Z}/2)^{r-1} \times (\mu_2)^r \to 1.$$

In this case, an argument analogous to the one for G shows that every representation of G_1 on which the center μ_2 acts by its standard representation has dimension a multiple of 2^{r-1} (rather than 2^r). The argument is otherwise identical to the argument for $\mathrm{Spin}(2r+1)$, and we find that $\mathrm{ed}_2(\mathrm{Spin}(2r)) \geq 2^{r-1} - r(2r-1)$. For r odd at least 9, this agrees with the lower bound found earlier, which proves the theorem on $\mathrm{Spin}(n)$ for $n \equiv 0 \pmod 4$.

It remains to show that for n a multiple of 4, with 2^m the largest power of 2 dividing n, we have

$$\mathrm{ed}_2(\mathrm{Spin}(n)) \geq 2^{(n-2)/2} + 2^m - n(n-1)/2.$$

The argument follows that of Merkurjev in characteristic not 2 [17, Theorem 4.9].

Namely, for n a multiple of 4, the center C of $G := \mathrm{Spin}(n)$ is isomorphic to $\mu_2 \times \mu_2$, and $H := G/C$ is the group $PGO^+(n)$. An H-torsor over a field L over k is equivalent to a central simple algebra A of degree n over L with a quadratic pair (σ, f) and with trivialized discriminant, meaning an isomorphism from the center of the Clifford algebra $C(A, \sigma, f)$ to $L \times L$ [13, section 29.F]. The image of the homomorphism from $C^* \cong (\mathbf{Z}/2)^2$ to the Brauer group of L is equal to $\{0, [A], [C^+], [C^-]\}$, where C^+ and C^- are the simple components of the Clifford algebra; each is a central simple algebra of degree $2^{(n-2)/2}$ over L. By Merkurjev, there is a field L over k and an H-torsor E over L such that $\mathrm{ind}(C^+) = \mathrm{ind}(C^-) = 2^{(n-2)/2}$ and $\mathrm{ind}(A) = 2^m$ [16, section 4.4 and Theorem 5.2]. We use the following result [17, Example 3.7]:

Lemma 2.3. *Let L be a field, p a prime number, and r a natural number. Let C be the group scheme $(\mu_p)^r$, and let Y be a C-gerbe over L. Then the p-essential dimension of Y, and also the essential dimension of Y, is the minimum, over all bases $u_1, \ldots, u_r$ for C^*, of $\sum_{i=1}^r \mathrm{ind}(u_i(Y))$.*

It follows that the 2-essential dimension of the $(\mu_2)^2$-gerbe E/G over L associated to the H-torsor E above is

$$\mathrm{ed}_2(E/G) = \mathrm{ind}(A) + \mathrm{ind}(C^+) = 2^{(n-2)/2} + 2^m.$$

It follows that

$$\begin{aligned}
\mathrm{ed}(\mathrm{Spin}(n)) &\geq \mathrm{ed}_2(\mathrm{Spin}(n)) \\
&\geq \mathrm{ed}_2(E/G) - \dim(G/C) \\
&= 2^{(n-2)/2} + 2^m - n(n-1)/2.
\end{aligned}$$

□

3 Etale motivic cohomology

In this section, we summarize the properties of etale motivic cohomology of fields, the natural home of mod p cohomological invariants for group schemes over a field of characteristic p.

For a field k of characteristic $p > 0$, let $H^{i,j}(k)$ be the etale motivic cohomology group $H^i_{\mathrm{et}}(k, \mathbf{Z}/p(j))$, or equivalently

$$H^i_{\mathrm{et}}(k, \mathbf{Z}/p(j)) \cong H^{i-j}_{\mathrm{et}}(k, \Omega^j_{\log}),$$

where $\Omega^j_{\log}$ is the subgroup of the group Ω^j of differential forms on the separable closure k_s over $\mathbf{F}_p$ spanned by products $(da_1/a_1) \wedge \cdots \wedge (da_j/a_j)$ with $a_1, \ldots, a_j \in k_s^*$ [9]. The group $H^{i,j}(k)$ is zero except when i equals j or $j+1$, because k has p-cohomological dimension at most 1 [21, section II.2.2]. The symbol $\{a_1, \ldots, a_{n-1}, b]$

denotes the element of $H^{n,n-1}(k)$ which is the product of the elements $a_i \in k^*/(k^*)^p \cong H^{1,1}(k)$ and $b \in k/\{a^p - a : a \in k\} \cong H^{1,0}(k)$.

Also, for a field k of characteristic 2, let $W(k)$ denote the Witt ring of symmetric bilinear forms over k, and let $I_q(k)$ be the Witt group of nondegenerate quadratic forms over k. (By the conventions in section 1, $I_q(k)$ consists only of even-dimensional forms.) Then $I_q(k)$ is a module over $W(k)$ via tensor product [5, Lemma 8.16]. Let I be the kernel of the homomorphism rank: $W(k) \to \mathbf{Z}/2$, and let

$$I_q^m(k) = I^{m-1} \cdot I_q(k),$$

following [5, p. 53]. To motivate the notation, observe that the class of an m-fold quadratic Pfister form $\langle\langle a_1, \ldots, a_{m-1}, b]]$ lies in $I_q^m(k)$. By definition, for $a_1, \ldots, a_{m-1}$ in k^* and b in k, $\langle\langle a_1, \ldots, a_{m-1}, b]]$ is the quadratic form $\langle\langle a_1\rangle\rangle_b \otimes \cdots \otimes \langle\langle a_{m-1}\rangle\rangle_b \otimes \langle\langle b]]$ of dimension 2^m, where $\langle\langle a\rangle\rangle_b$ is the bilinear form $\langle 1, a\rangle$ and $\langle\langle b]]$ is the quadratic form $[1, b] = x^2 + xy + by^2$.

In analogy with the Milnor conjecture, Kato proved the isomorphism

$$I_q^m(F)/I_q^{m+1} \cong H^{m,m-1}(F)$$

for every field F of characteristic 2 [5, Fact 16.2]. The isomorphism takes the quadratic Pfister form $\langle\langle a_1, \ldots, a_{m-1}, b]]$ to the symbol $\{a_1, \ldots, a_{m-1}, b]$. (For this paper, it would suffice to have Kato's homomorphism, without knowing that it is an isomorphism.)

A cohomological invariant gives a lower bound for the essential dimension, as follows. This is standard for mod l invariants with $l \neq p = \text{char}(k)$ [17, Theorem 5.3], and we now give the analogous statement for mod p invariants of a k-group scheme G. Define a cohomological invariant f of G with values in $H^{n,n-1}$ to be *nontrivial* if there is a field F containing an algebraic closure of k and a G-torsor u over F such that $f(u)$ is not zero.

Lemma 3.1. *Let G be an affine group scheme of finite type over a field k of characteristic $p > 0$. If there is a nontrivial cohomological invariant for G with values in $H^{n,n-1}$, then* $\text{ed}(G) \geq \text{ed}_p(G) \geq n$.

Proof. Let f be the given cohomological invariant for G. It suffices to prove a lower bound on the essential dimension after enlarging k. So we can replace k by its algebraic closure. Then every field F of transcendence degree less than n over k has $H^{n,n-1}(F) = 0$, by Kato and Kuzumaki [12, section 3, Corollary 2]. By assumption, there is a G-torsor u over a field E over k such that $f(u)$ is not zero in $H^{n,n-1}(E)$. Thanks to the transfer maps on Galois cohomology (viewing $H^{n,n-1}(E)$ as $H^1(E, \Omega^{n-1}_{\log}(E_s)))$, this element remains nonzero in $H^{n,n-1}(E')$ for any finite extension E'/E of degree prime to p. Therefore, the G-torsor u extended up to E' cannot be defined over a subfield F of E' with transcendence degree less than n over k. So $\text{ed}(G) \geq \text{ed}_p(G) \geq n$. ☐

Corollary 3.2. *Let G be an affine group scheme of finite type over a field k of characteristic $p > 0$. Let f be a cohomological invariant for G with values in $H^{n,n-1}$. Suppose that for any field F over k and any $a_1, \ldots, a_{n-1}$ in F^* and a_n in F, there is a G-torsor u over F with*

$$f(u) = \{a_1, \ldots, a_{n-1}, a_n]$$

in $H^{n,n-1}(F)$. *Then* $\mathrm{ed}(G) \geq \mathrm{ed}_p(G) \geq n$.

Proof. Let $\overline{k}$ be an algebraic closure of k, and let E be the rational function field $\overline{k}(a_1, \ldots, a_n)$. By assumption, there is a G-torsor u over E such that

$$f(u) = \{a_1, \ldots, a_{n-1}, a_n].$$

This symbol in $H^{n,n-1}(E)$ is not zero, by Izhboldin's calculation of $H^{n,n-1}$ of a rational function field [10, Theorem 4.5]. Thus f is nontrivial, in the sense above. By Lemma 3.1, $\mathrm{ed}(G) \geq \mathrm{ed}_p(G) \geq n$. □

4 Low-dimensional spin groups

Rost and Garibaldi determined the essential dimension of the spin groups $\mathrm{Spin}(n)$ with $n \leq 14$ in characteristic not 2 [6, Table 23B]. It should be possible to compute the essential dimension of low-dimensional spin groups in characteristic 2 as well. The following section carries this out for $\mathrm{Spin}(n)$ with $n \leq 10$. We find that in this range (as for $n \geq 15$), the essential dimension of the spin group is the same in characteristic 2 as in characteristic not 2, unlike what happens for $O(n)$ and $SO(n)$.

For $n \leq 10$, we give group-theoretic proofs which work almost the same way in any characteristic, despite the distinctive features of quadratic forms in characteristic 2.

Theorem 4.1. *For* $n \leq 10$, *the essential dimension, as well as the 2-essential dimension, of the split group* $\mathrm{Spin}(n)$ *over a field* k *of any characteristic is given by:*

n	$\mathrm{ed}(\mathrm{Spin}(n))$
≤ 6	0
7	4
8	5
9	5
10	4

Proof. As discussed above, it suffices to consider the case of a field k of characteristic 2. For $2 \leq n \leq 6$, every $\mathrm{Spin}(n)$-torsor over a field is trivial, for example by the exceptional isomorphisms $\mathrm{Spin}(2) \cong G_m$, $\mathrm{Spin}(3) \cong SL(2)$, $\mathrm{Spin}(4) \cong SL(2) \times SL(2)$, $\mathrm{Spin}(5) \cong Sp(4)$, and $\mathrm{Spin}(6) \cong SL(4)$. It follows that $\mathrm{ed}(\mathrm{Spin}(n)) = 0$ for $2 \leq n \leq 6$.

We will use the following standard approach to bounding the essential dimension of a group.

Lemma 4.2. *Let G be an affine group scheme of finite type over a field k. Suppose that G acts on a k-scheme Y with a nonempty open orbit U. Suppose that for every G-torsor E over an infinite field F over k, the twisted form $(E \times Y)/G$ of Y over F has a Zariski-dense set of F-points. Finally, suppose that U has a k-point x, and let N be the stabilizer k-group scheme of x in G. Then*

$$H^1(F, N) \to H^1(F, G)$$

is surjective for every infinite field F over k (or for every field F over k, if G is smooth and connected). As a result, $\mathrm{ed}_k(G) \leq \mathrm{ed}_k(N)$.

n	char $k \neq 2$	char $k = 2$
6	$SL(3) \cdot (G_a)^3$	same
7	G_2	same
8	Spin(7)	same
9	Spin(7)	same
10	$\text{Spin}(7) \cdot (G_a)^8$	same
11	$SL(5)$	$\mathbf{Z}/2 \ltimes SL(5)$
12	$SL(6)$	$\mathbf{Z}/2 \ltimes SL(6)$
13	$SL(3) \times SL(3)$	$\mathbf{Z}/2 \ltimes (SL(3) \times SL(3))$
14	$G_2 \times G_2$	$\mathbf{Z}/2 \ltimes (G_2 \times G_2)$

Table 1: Generic stabilizer of spin (or half-spin) representation of Spin(n)

The proof is short, the same as that of [6, Theorem 9.3]. (Note that even if k is finite, we get the stated upper bound for the essential dimension of G: a G-torsor over a finite field F that contains k causes no problem, because F has transcendence degree 0 over k.) If G is smooth and connected, then $H^1(F, G)$ is in fact trivial for every finite field F that contains k, by Lang [14]; that implies the statement in the theorem that $H^1(F, N) \to H^1(F, G)$ is surjective for *every* field F over k.

The assumption about a Zariski-dense set of rational points holds, for example, if Y is a linear representation V of G, or if Y is the associated projective space $P(V)$ to a representation, or (as we use later) a product $P(V) \times P(W)$.

We use Garibaldi and Guralnick's calculation of the stabilizer group scheme of a general k-point in the spin (for n odd) or a half-spin (for n even) representation W of the split group Spin(n), listed in Table 1 here [7, Table 1]. Here Spin(n) has an open orbit on the projective space $P(W)$ of lines in W if $n \leq 12$ or $n = 14$, and an open orbit on W if $2 \leq n \leq 6$ or $n = 10$. (To be precise, we will use that even if k is finite, there is a k-point in the open orbit for which the stabilizer k-group scheme is the *split* group listed in the table.)

We now begin to compute the essential dimension of the split group $G = \text{Spin}(7)$ over a field k of characteristic 2. Let W be the 8-dimensional spin representation of G. Then G has an open orbit on the projective space $P(W)$ of lines in W. By Table 1, there is a k-point x in W whose image in $P(W)$ is in the open orbit such that the stabilizer of x in G is the split exceptional group G_2. Since G preserves a quadratic form on W, the stabilizer H of the corresponding k-point in $P(W)$ is at most $G_2 \times \mu_2$. In fact, H is equal to $G_2 \times \mu_2$, because the center μ_2 of G acts trivially on $P(W)$.

By Lemma 4.2, the inclusion $G_2 \times \mu_2 \hookrightarrow G$ induces a surjection

$$H^1(F, G_2 \times \mu_2) \to H^1(F, G)$$

for every field F over k. Over any field F, G_2-torsors up to isomorphism can be identified with 3-fold quadratic Pfister forms $\langle\langle a_1, a_2, b]]$ (with $a_1, a_2 \in F^*$ and $b \in F$), and so G_2 has essential dimension 3 [21, Théorème 11]. Since μ_2 has essential dimension 1, the surjectivity above implies that $G = \text{Spin}(7)$ has essential dimension at most 4.

Next, a G-torsor determines two quadratic forms of dimension 8. Besides the obvious homomorphism $\chi_1 \colon G \hookrightarrow \text{Spin}(8) \to SO(8)$ (which is trivial on the center μ_2

of G), we have the spin representation $\chi_2 \colon G \to SO(8)$, on which μ_2 acts faithfully by scalars. Thus a G-torsor u over a field F over k determines two quadratic forms of dimension 8 over F, which we call q_1 and q_2.

To describe these quadratic forms in more detail, use that every G-torsor comes from a torsor for $G_2 \times \mu_2$. The two homomorphisms $G_2 \hookrightarrow G \to SO(8)$ (via χ_1 and χ_2) are both conjugate to the standard inclusion. Also, χ_1 is trivial on the μ_2 factor, while χ_2 acts faithfully by scalars on the μ_2 factor. It follows that q_1 is a quadratic Pfister form, $\langle\langle a, b, c]]$ (the form associated to a G_2-torsor), while q_2 is a scalar multiple of that form, $d\langle\langle a, b, c]]$.

Therefore, a G-torsor u canonically determines a 4-fold quadratic Pfister form,

$$q_1 + q_2 = \langle\langle d, a, b, c]].$$

Define $f_4(u)$ to be the associated element of $H^{4,3}(F)$,

$$f_4(u) = \{d, a, b, c].$$

By construction, this is well-defined and an invariant of u. By considering the subgroup $G_2 \times \mu_2 \subset \mathrm{Spin}(7)$, where there is a $G_2 \times \mu_2$-torsor associated to any elements a, b, d in F^* and c in F, we see that a, b, c, d can be chosen arbitrarily. By Corollary 3.2, $G = \mathrm{Spin}(7)$ has 2-essential dimension at least 4, and hence essential dimension at least 4.

The opposite inequality was proved above, and so Spin(7) has essential dimension equal to 4. Since the lower bound is proved by constructing a mod 2 cohomological invariant, this argument also shows that Spin(7) has 2-essential dimension equal to 4. For the same reason, the computations of essential dimension below (for $\mathrm{Spin}(n)$ with $8 \le n \le 10$) also give the 2-essential dimension.

Next, we turn to Spin(8). At first, let $G = \mathrm{Spin}(2r)$ for a positive integer r over a field k of characteristic 2. Let V be the standard $2r$-dimensional representation of G. Then G has an open orbit in the projective space $P(V)$ of lines in V. The stabilizer k-group scheme H of a general k-point in $P(V)$ is conjugate to $\mathrm{Spin}(2r-1)\cdot Z$, where Z is the center of $\mathrm{Spin}(2r)$, with $\mathrm{Spin}(2r-1) \cap Z = \mu_2$. (In more detail, a general line in V is spanned by a vector x with $q(x) \neq 0$, where q is the quadratic form on V. Then the stabilizer of x in $SO(V)$ is isomorphic to $SO(S)$, where $S := x^\perp$ is a hyperplane in V on which q restricts to a nonsingular quadratic form of dimension $2r - 1$, with $S^\perp$ equal to the line $k \cdot x \subset S$.) Here

$$Z \cong \begin{cases} \mu_2 \times \mu_2 & \text{if } r \text{ is even} \\ \mu_4 & \text{if } r \text{ is odd.} \end{cases}$$

In particular, if r is even, then $H \cong \mathrm{Spin}(2r-1) \times \mu_2$. Thus, for r even, the inclusion $\mathrm{Spin}(2r-1) \times \mu_2 \hookrightarrow G$ induces a surjection

$$H^1(F, \mathrm{Spin}(2r-1) \times \mu_2) \to H^1(F, G)$$

for every field F over k, by Lemma 4.2.

It follows that, for r even, the essential dimension of $\mathrm{Spin}(2r)$ is at most 1 plus the essential dimension of $\mathrm{Spin}(2r-1)$. Since Spin(7) has essential dimension 4, $G = \mathrm{Spin}(8)$ has essential dimension at most 5.

Before proving that equality holds, let us analyze G-torsors in more detail. We know that $H^1(F, \text{Spin}(7) \times \mu_2) \to H^1(F, G)$ is onto, for all fields F over k. Also, we showed earlier that $H^1(F, G_2 \times \mu_2) \to H^1(F, \text{Spin}(7))$ is surjective. Therefore,

$$H^1(F, G_2 \times \mu_2 \times \mu_2) \to H^1(F, G)$$

is surjective for all fields F over k, where $Z = \mu_2 \times \mu_2$ is the center of G. As discussed earlier, G_2-torsors up to isomorphism can be identified with 3-fold quadratic Pfister forms. It follows that every G-torsor is associated to some 3-fold quadratic Pfister form $\langle\langle a, b, c]]$ and some elements d, e in F^*, which yield elements of $H^1(F, \mu_2) = F^*/(F^*)^2$.

Next, observe that a G-torsor determines several quadratic forms. Besides the obvious double covering $\chi_1 \colon G \to SO(8)$, the two half-spin representations of G give two other homomorphisms $\chi_2, \chi_3 \colon G \to SO(8)$. (These three homomorphisms can be viewed as the quotients of G by the three k-subgroup schemes of order 2 in Z. They are permuted by the group S_3 of "triality" automorphisms of G.) Thus a G-torsor u over a field F over k determines three quadratic forms of dimension 8, which we call q_1, q_2, q_3.

To describe how these three quadratic forms are related, use that every G-torsor comes from a torsor for $G_2 \times \mu_2 \times \mu_2$. The three homomorphisms $G_2 \to G \to SO(8)$ (via χ_1, χ_2, and χ_3) are all conjugate to the standard inclusion, whereas the three homomorphisms send $\mu_2 \times \mu_2$ to the center $\mu_2 \subset SO(8)$ by the three possible surjections. It follows that the three quadratic forms can be written as $q_1 = d\langle\langle a, b, c]]$, $q_2 = e\langle\langle a, b, c]]$, and $q_3 = de\langle\langle a, b, c]]$.

Note that a scalar multiple of a quadratic Pfister form, $q = d\langle\langle a_1, \ldots, a_{m-1}, b]]$ (as a quadratic form up to isomorphism), uniquely determines the associated quadratic Pfister form $q_0 = \langle\langle a_1, \ldots, a_{m-1}, b]]$ up to isomorphism. (Proof: it suffices to show that if q and r are m-fold quadratic Pfister forms over F with $aq \cong r$ for some a in F^*, then $q \cong r$. Since r takes value 1, so does aq, and so q takes value a^{-1}. But then $a^{-1}q \cong q$ by the multiplicativity of quadratic Pfister forms [5, Corollary 9.9]. Therefore, $r \cong aq \cong q$.)

We now define an invariant for $G = \text{Spin}(8)$ over k with values in $H^{5,4}$. Given a G-torsor u over a field F over k, consider the three associated quadratic forms q_1, q_2, q_3 as above. By the previous paragraph, $q_1 = d\langle\langle a, b, c]]$ determines the quadratic Pfister form $q_0 = \langle\langle a, b, c]]$. So u determines the 5-fold quadratic Pfister form

$$q_0 + q_1 + q_2 + q_3 = \langle\langle d, e, a, b, c]].$$

The associated class

$$f_5(u) = \{d, e, a, b, c] \in H^{5,4}(F)$$

is therefore an invariant of u.

By considering the subgroup $G_2 \times Z \subset G = \text{Spin}(8)$, where $Z = \mu_2 \times \mu_2$, there is a $G_2 \times Z$-torsor associated to any elements a, b, d, e in F^* and c in F, and f_5 of the associated G-torsor is $\{d, e, a, b, c]$ in $H^{5,4}(F)$. By Corollary 3.2, G has essential dimension at least 5. Since the opposite inequality was proved above, $G = \text{Spin}(8)$ has essential dimension over k equal to 5.

Next, let $G = \text{Spin}(9)$ over a field k of characteristic 2. Let W be the spin representation of G, of dimension 16, corresponding to a homomorphism $G \to$

$SO(16)$. (A reference for the fact that this self-dual representation is orthogonal in characteristic 2, as in other characteristics, is [8, Theorem 9.2.2].) By Table 1, G has an open orbit on the space $P(W)$ of lines in W, and the stabilizer in G of a general k-point in W is conjugate to Spin(7). (This is not the standard inclusion of Spin(7) in Spin(9), but rather a lift of the spin representation $\chi_2\colon \text{Spin}(7) \to SO(8)$ to Spin(8) followed by the standard inclusion Spin(8) $\hookrightarrow$ Spin(9). In particular, the image of Spin(7) does not contain the center μ_2 of $G = \text{Spin}(9)$.) Since G preserves a quadratic form on W, it follows that the stabilizer in G of a general k-point in $P(W)$ is conjugate to $\text{Spin}(7) \times \mu_2$, where μ_2 is the center of Spin(9) (which acts faithfully by scalars on W). Therefore, by Lemma 4.2, the inclusion of $\text{Spin}(7) \times \mu_2$ in $G = \text{Spin}(9)$ induces a surjection

$$H^1(F, \text{Spin}(7) \times \mu_2) \to H^1(F, G)$$

for every field F over k.

Since Spin(7) has essential dimension 4 over k as shown above, $G = \text{Spin}(9)$ has essential dimension at most $4 + 1 = 5$.

Next, a G-torsor determines several quadratic forms. Besides the obvious homomorphism $R\colon G \hookrightarrow \text{Spin}(10) \to SO(10)$, we have the spin representation $S\colon G \to SO(16)$. Thus a G-torsor over a field F over k determines a quadratic form r of dimension 10 and a quadratic form s of dimension 16.

To describe how these forms are related, use that every G-torsor comes from a torsor for the subgroup $\text{Spin}(7) \times \mu_2$ described above. The restriction of R to the given subgroup Spin(7) is the composition of the spin representation $\chi_2\colon \text{Spin}(7) \to SO(8)$ with the obvious inclusion $SO(8) \hookrightarrow SO(10)$. The restriction of S to the given subgroup Spin(7) is the direct sum of the standard representation $\chi_1\colon \text{Spin}(7) \to SO(8)$ and the spin representation $\chi_2\colon \text{Spin}(7) \to SO(8)$. Finally, R is trivial on the second factor μ_2 (the center of G), whereas S acts faithfully by scalars on S.

Now, let (u_1, e) be a $\text{Spin}(7) \times \mu_2$-torsor over k, where u_1 is a Spin(7)-torsor and e is in $H^1(F, \mu_2) = F^*/(F^*)^2$, which we lift to an element e of F^*. By the earlier analysis of the quadratic forms associated to a Spin(7)-torsor, the quadratic form associated to u_1 via the standard representation $\chi_1\colon \text{Spin}(7) \to SO(8)$ is a 3-fold quadratic Pfister form $\langle\langle a, b, c]]$, while the quadratic form associated to u_1 via the spin representation $\chi_2\colon \text{Spin}(7) \to SO(8)$ is a multiple of the same form, $d\langle\langle a, b, c]]$.

By the analysis of representations two paragraphs back, it follows that the quadratic form associated to (u_1, e) via the representation $R\colon G \to SO(10)$ is $r = H + d\langle\langle a, b, c]]$, where H is the hyperbolic plane. Also, the quadratic form associated to (u_1, e) via the representation $S\colon G \to SO(16)$ is $s = e\langle\langle a, b, c]] + de\langle\langle a, b, c]]$.

Next, r determines the quadratic form $r_0 = d\langle\langle a, b, c]]$ by Witt cancellation [5, Theorem 8.4], and that in turn determines the quadratic Pfister form $q_0 = \langle\langle a, b, c]]$ as shown above. Therefore, a G-torsor u determines the 5-fold quadratic Pfister form

$$q_0 + r_0 + s = \langle\langle d, e, a, b, c]]$$

up to isomorphism.

Therefore, defining

$$f_5(u) = \{d, e, a, b, c]$$

in $H^{5,4}(F)$ yields an invariant of u. By our earlier description of Spin(7)-torsors, we can take a, b, d, e to be any elements of F^* and c any element of F. By Corollary

3.2, G has essential dimension at least 5. Since the opposite inequality was proved earlier, $G = \text{Spin}(9)$ over k has essential dimension equal to 5.

Finally, let $G = \text{Spin}(10)$ over a field k of characteristic 2. Let V be the 10-dimensional standard representation of G, corresponding to the double covering $G \to SO(10)$, and let W be one of the 16-dimensional half-spin representations of G, corresponding to a homomorphism $G \to SL(16)$. (The other half-spin representation of G is the dual W^*.)

As discussed above for any group $\text{Spin}(2r)$, $G = \text{Spin}(10)$ has an open orbit on $P(V)$, with generic stabilizer $\text{Spin}(9)\cdot\mu_4$. (Here μ_4 is the center of G, which contains the center μ_2 of Spin(9).) Consider the action of G on $P(V) \times P(W) \cong \mathbf{P}^9 \times \mathbf{P}^{15}$. As discussed above, Spin(9) (and hence $\text{Spin}(9) \cdot \mu_4$) has an open orbit on $P(W)$. As a result, G has an open orbit on $P(V) \times P(W)$. Moreover, the generic stabilizer of Spin(9) on $P(W)$ is $\text{Spin}(7) \times \mu_2$, where the inclusion $\text{Spin}(7) \hookrightarrow \text{Spin}(9)$ is the composition of the spin representation $\text{Spin}(7) \hookrightarrow \text{Spin}(8)$ with the standard inclusion into Spin(9); in particular, the image does not contain the center μ_2 of Spin(9). Therefore, the generic stabilizer of $\text{Spin}(9) \cdot \mu_4 \subset \text{Spin}(10)$ on $P(W)$ is $\text{Spin}(7) \times \mu_4$. We conclude that G has an open orbit on $P(V) \times P(W)$, with generic stabilizer $\text{Spin}(7) \times \mu_4$. It follows that

$$H^1(F, \text{Spin}(7) \times \mu_4) \to H^1(F, G)$$

is surjective for every field F over k, by Lemma 4.2.

The image H_2 of the subgroup $H = \text{Spin}(7) \times \mu_4 \subset G$ in $SO(10)$ is $\text{Spin}(7) \times \mu_2$, where Spin(7) is contained in $SO(8)$ (and contains the center μ_2 of $SO(8)$) and μ_2 is the center of $SO(10)$. In terms of the subgroup $SO(8) \times SO(2)$ of $SO(10)$, we can also describe H_2 as $\text{Spin}(7) \times \mu_2$, where Spin(7) is contained in $SO(8)$ and μ_2 is contained in $SO(2)$. Thus H_2 is contained in $\text{Spin}(7) \times SO(2)$. Therefore, H is contained in $\text{Spin}(7) \times G_m \subset G = \text{Spin}(10)$, where G_m is the inverse image in G of $SO(2) \subset SO(10)$. It follows that

$$H^1(F, \text{Spin}(7) \times G_m) \to H^1(F, G)$$

is surjective for every field F over k. Since every G_m-torsor over a field is trivial,

$$H^1(F, \text{Spin}(7)) \to H^1(F, G)$$

is surjective for every field F over k.

Here Spin(7) maps into Spin(8) by the spin representation, and then $\text{Spin}(8) \hookrightarrow G = \text{Spin}(10)$ by the standard inclusion. By the description above of the 8-dimensional quadratic form associated to a Spin(7)-torsor by the spin representation, it follows that the quadratic form associated to a G-torsor is of the form $H + d\langle\langle a, b, c]]$.

Every 10-dimensional quadratic form in I_q^3 over a field is associated to some G-torsor. So we have given another proof that every 10-dimensional quadratic form in I_q^3 is isotropic. This was proved in characteristic not 2 by Pfister, and it was extended to characteristic 2 by Baeza and Tits, independently [2, pp. 129-130], [22, Theorem 4.4.1(ii)].

Since Spin(7) has essential dimension 4, the surjectivity above implies that $G = \text{Spin}(10)$ has essential dimension at most 4. To prove equality, we define an invariant for G with values in $H^{4,3}$ by the same argument used for Spin(7). Namely, a G-torsor u over a field F over k determines a 4-fold quadratic Pfister form

$$\langle\langle d, a, b, c]]$$

up to isomorphism, and hence the element

$$f_4(u) = \{d, a, b, c]$$

in $H^{4,3}(F)$. By Corollary 3.2, this completes the proof that $G = \mathrm{Spin}(10)$ over k has essential dimension equal to 4. As in the previous cases, since the lower bound is proved using a mod 2 cohomological invariant, G also has 2-essential dimension equal to 4. □

References

[1] A. Babic and V. Chernousov. Lower bounds for essential dimensions in characteristic 2 via orthogonal representations. *Pac. J. Math.* **279** (2015), 36–63. 1, 3

[2] R. Baeza. *Quadratic forms over semilocal rings.* Lecture Notes in Mathematics 655, Springer (1978). 14

[3] P. Brosnan, Z. Reichstein, and A. Vistoli. Essential dimension, spinor groups and quadratic forms. *Ann. Math.* **171** (2010), 533–544. 1, 2

[4] V. Chernousov and A. Merkurjev. Essential dimension of spinor and Clifford groups. *Algebra Number Theory* **2** (2014), 457–472. 1, 3, 4

[5] R. Elman, N. Karpenko, and A. Merkurjev. *The algebraic and geometric theory of quadratic forms.* Amer. Math. Soc. (2008). 8, 12, 13

[6] S. Garibaldi. Cohomological invariants: exceptional groups and spin groups. *Mem. Amer. Math. Soc.* **200** (2009), no. 937. 1, 9, 10

[7] S. Garibaldi and R. Guralnick. Spinors and essential dimension. *Compos. Math.,* to appear. 1, 3, 10

[8] S. Garibaldi and D. Nakano. Bilinear and quadratic forms on rational modules of split reductive groups. *Canad. J. Math.* **68** (2016), 395–421. 13

[9] T. Geisser and M. Levine. The K-theory of fields in characteristic p. *Invent. Math.* **139** (2000), 459–493. 7

[10] O. Izhboldin. On the cohomology groups of the field of rational functions. *Mathematics in St. Petersburg*, 21–44, Amer. Math. Soc. Transl. Ser. 2, 174, Amer. Math. Soc. (1996). 9

[11] N. Karpenko and A. Merkurjev. Essential dimension of finite p-groups. *Invent. Math.* **172** (2008), 491–508. 4, 6

[12] K. Kato and T. Kuzumaki. The dimension of fields and algebraic K-theory. *J. Number Theory* **24** (1986), 229–244. 8

[13] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol. *The book of involutions.* Amer. Math. Soc. (1998). 2, 4, 7

[14] S. Lang. Algebraic groups over finite fields. *Amer. J. Math.* **78** (1956), 555–563. 10

[15] R. Lötscher. A fiber dimension theorem for essential and canonical dimension. *Compos. Math.* **149** (2013), 148–174. 4

[16] A. Merkurjev. Maximal indexes of Tits algebras. *Doc. Math.* **1** (1996), 229–243. 6, 7

[17] A. Merkurjev. Essential dimension. *Quadratic forms – algebra, arithmetic, and geometry*, 299–325, Contemp. Math., 493, Amer. Math. Soc. (2009). 1, 6, 7, 8

[18] A. Merkurjev. Essential dimension. *Bull. Amer. Math. Soc.*, to appear. 1, 2

[19] Z. Reichstein. Essential dimension. *Proceedings of the International Congress of Mathematicians*, v. II, 162–188. Hindustan Book Agency, New Delhi (2010). 1, 2

[20] T. Sekiguchi. On projective normality of abelian varieties. II. *J. Math. Soc. Japan* **29** (1977), 709–727. 5

[21] J.-P. Serre. *Galois cohomology.* Springer (2002). 7, 10

[22] J. Tits. Strongly inner anisotropic forms of simple algebraic groups. *J. Algebra* **131** (1990), 648–677. 14

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