The automorphism group of an affine quadric

Burt Totaro

We determine the automorphism group for a large class of affine quadrics over a field, viewed as affine algebraic varieties. The proof uses a fundamental theorem of Karpenko's in the theory of quadratic forms [13], along with some useful arguments of birational geometry. In particular, we find that the automorphism group of the n-sphere $\{x_0^2 + \cdots + x_n^2 = 1\}$ over the real numbers is just the orthogonal group $O(n+1)$ whenever n is a power of 2. It is not known whether the same is true for arbitrary n. This result is reminiscent of Wood's theorem that when n is a power of 2, every real polynomial mapping from the n-sphere to a lower-dimensional sphere is constant [22].

The background for these results is that almost all geometric tools work better for projective varieties than for affine varieties, because of the lack of compactness. Even basic questions like determining the automorphism group of an affine variety, or whether two affine varieties are isomorphic, can be difficult. Of course, some cases are easy. Consider an affine variety $X - D$ where X is projective. If X is of general type, or more generally if the pair (X, D) is of log-general type in Iitaka's sense (for example, when D is a smooth hypersurface of degree at least $n+2$ in $X = \mathbf{P}^n$), then the affine variety $X - D$ has finite automorphism group [10, Theorem 11.12]. But when both X and D are of low degree in some sense, then the automorphism group of $X - D$ is not at all understood. This justifies studying the basic case of affine quadrics.

The key ingredient of the proof of the main Theorem 2.3 is that an anisotropic projective quadric with first Witt index equal to 1 is not ruled. More generally, a fundamental problem of birational geometry is to determine which varieties over a field are ruled. For example, Kollár proved that a large class of rationally connected complex hypersurfaces are non-rational by showing that they are not even ruled [16, Theorem V.5.14]. For anisotropic quadrics over a field, we give a conjectural answer to the problem of ruledness: they should be ruled if and only if the first Witt index is greater than 1 (Conjecture 3.1). Section 3 gives some evidence: in particular, the conjecture is true for quadratic forms of dimension at most 9 (thus for projective quadrics of dimension at most 7).

1 Infinite-dimensional automorphism groups

In this section, we show that all affine quadrics of dimension at least 2 whose homogeneous part of degree 2 is isotropic have infinite-dimensional automorphism group. Here an affine quadric is the affine scheme defined by the vanishing of a polynomial f of degree at most 2 over a field k. We assume only that f is not a constant; we do not assume any nondegeneracy, and the field k may be of any characteristic including 2. A quadratic form is defined to be a homogeneous polynomial of degree 2 on a finite-dimensional vector space V over a field k. We say that a quadratic form q is isotropic if there is a nonzero vector x in V with $q(x) = 0$.

Lemma 1.1 *Let $X = \{f = 0\}$ be an affine quadric in an n-dimensional vector space over a field k with $n \geq 3$. If the homogeneous part of degree 2 of f is isotropic, then the automorphism group of X is infinite-dimensional. Also, for any isotropic quadratic form q of dimension $n \geq 3$ over a field k, the complement $\mathbf{P}^{n-1} - \{q = 0\}$ has infinite-dimensional automorphism group.*

When $n = 3$, more is known. For an isotropic affine quadric surface $xy + az^2 = b$ over a field, or the complement of an isotropic conic $xy + az^2 = 0$ in $\mathbf{P}^2$, Gizatullin and Danilov computed the automorphism groups explicitly, as amalgamated free products [6].

For the purposes of this section, we will say that the automorphism group of an algebraic variety X over a field k is infinite-dimensional if there are unirational varieties S over k of arbitrarily large dimension which have a morphism $S \times X \to X$ that induces an injection of $S(k)$ into $\mathrm{Aut}(X)$. (Here unirationality, being the image of a dominant rational map from projective space, ensures that the set $S(k)$ of k-rational points is "big", at least for infinite fields k.) One could consider weaker definitions, but Lemma 1.1 will show that certain affine quadrics have infinite-dimensional automorphism group over k in this strong sense.

Proof. We can view X as a projective quadric minus its intersection with a hyperplane. Let the projective quadric be defined by a quadratic form q on a vector space V, while the hyperplane corresponds to a hyperplane $W \subset V$. We are assuming that q restricted to W is isotropic, and so we can choose a nonzero vector x in W with $q(x) = 0$. The main point is to exhibit a nontrivial homomorphism from the additive group G_a to the orthogonal group $O(W \subset V)$ of linear automorphisms of V that preserve q and the subspace W.

Let $\langle u, v\rangle = q(u + v) - q(u) - q(v)$ be the symmetric bilinear form associated to q. If the isotropic vector x in W is orthogonal to all of V, we choose a nonzero linear form $g : V \to k$ that vanishes on x. Then

$$\varphi_t(z) = z + tg(z)x$$

is a nontrivial homomorphism from G_a to $O(W \subset V)$. Next, suppose x is not orthogonal to all of V. Let y be a vector in W orthogonal to x and not a multiple of x; this exists, since W has dimension at least 3. Then the Siegel transvection [3, III.1.5]

$$\varphi_t(z) = z + t\langle z, x\rangle y - t\langle z, y\rangle x - t^2 q(y)\langle z, x\rangle x$$

is a nontrivial homomorphism from G_a to $O(W \subset V)$.

Thus, we have a nontrivial action of G_a on the affine quadric X. This gives an injective homomorphism from the k-vector space of G_a-invariant regular functions f on X to the automorphism group of X, given by $z \mapsto \varphi_{f(z)}(z)$. Since X has dimension at least 2, we see by hand that the vector space of G_a-invariant functions on X is infinite-dimensional (use arbitrary polynomials in the coordinates not changed by the G_a-action). Thus X has infinite-dimensional automorphism group. The same proof works for the complement $\mathbf{P}^{n-1} - \{q = 0\}$ of an isotropic quadratic form q, since there is a nontrivial homomorphism from G_a to the orthogonal group of q. QED

In particular, every quadratic form of dimension at least 2 over an algebraically closed field is isotropic. For example, Lemma 1.1 says that the sphere $x_0^2+\cdots+x_n^2 = 1$, viewed as an affine variety over the complex numbers, has infinite-dimensional automorphism group for n at least 2. In the simplest case $n=2$, we can change variables over the complex numbers to $x_1x_2+x_3^2=1$, and then the proof of Lemma 1.1 gives the well-known automorphisms

$$(x_1,x_2,x_3)\mapsto(x_1-f(x_2)^2x_2-2f(x_2)x_3,x_2,f(x_2)x_2+x_3)$$

for any polynomial f in one variable.

For affine quadrics whose homogeneous part is anisotropic, it is not known whether the automorphism group of X as an affine variety is just the orthogonal group; Theorem 2.3 proves this in many cases. By contrast, we now show that the group of birational automorphisms of a smooth quadric Q of dimension at least 2 is always infinite-dimensional. (We define this to mean that there are unirational families of arbitrarily large dimension over k of birational automorphisms of Q, as in the above definition for biregular automorphisms.)

Lemma 1.2 *Let Q be a smooth projective quadric of dimension at least 2 over a field k. Then the group of birational automorphisms of Q over k is infinite-dimensional.*

Proof. Consider a general linear projection from the quadric $Q^n\subset \mathbf{P}^{n+1}$ onto a linear space $\mathbf{P}^{n-r}$, for any $1\leq r\leq n-1$. This gives a rational map f from Q^n to $\mathbf{P}^{n-r}$ of which the general fiber is an r-dimensional quadric. This general fiber comes from a quadratic form ρ of dimension $r+2$ over the function field $l=k(x_1,\ldots,x_{n-r})$ of $\mathbf{P}^{n-r}$. The group of birational automorphisms of Q contains the group of birational automorphisms of Q over $\mathbf{P}^{n-r}$, which in turn contains the orthogonal group $(PGO(\rho))(l)$, using that $r\geq 1$. (The definition of PGO is recalled in section 2.) The identity component of the algebraic group $PGO(\rho)$ is reductive, and so it is unirational over l [4, Theorem 18.2]; also, it has positive dimension over $l=k(x_1,\ldots,x_{n-r})$. Thus, the group $(PGO(\rho))(l)$ gives an infinite-dimensional group of birational automorphisms of Q over k. QED

2 Finite-dimensional automorphism groups

We now present some useful general results on isomorphisms between open varieties. The proofs are easy consequences of Abhyankar's lemma in birational geometry. As an application, for any affine quadric whose homogeneous part of degree 2 is anisotropic with first Witt index equal to 1, the automorphism group is essentially the orthogonal group. The application uses Karpenko's theorem that a projective quadric with first Witt index equal to 1 is not ruled.

Abhyankar's lemma is the following statement [1, Proposition 4]; see Kollár [16, Theorem VI.1.2] for a recent exposition.

Lemma 2.1 *Let $\pi : Y \to X$ be a proper birational morphism of irreducible schemes, with Y normal and X regular. Then every exceptional divisor of π is ruled over its image. That is, if E is an irreducible divisor in Y which does not map birationally to its image E' in X, then E is birational over E' to a scheme $W \times_{E'} \mathbf{P}^1_{E'}$.*

We deduce the following information about isomorphisms between open varieties, of which part (2) is especially powerful. A variety over a field k is *ruled* if it is birational to $Y \times \mathbf{P}^1$ for some variety Y over k.

Theorem 2.2 *(1) Let (X_1, D_1) and (X_2, D_2) be pairs with X_i a projective variety over a field k and D_i an irreducible divisor, for $i = 1$ and 2. We assume that each X_i is regular in a neighborhood of D_i. Suppose that D_1 is not ruled over k. Then any isomorphism f from $X_1 - D_1$ to $X_2 - D_2$, viewed as a birational map from X_1 to X_2, is an isomorphism in codimension 1. That is, there are subsets S_i of codimension at least 2 in X_i such that f is an isomorphism from $X_1 - S_1$ to $X_2 - S_2$.*

(2) Suppose in addition that the divisors D_1 and D_2 are ample. Then any isomorphism from $X_1 - D_1$ to $X_2 - D_2$ extends to an isomorphism from X_1 to X_2.

Under the stronger assumptions that D_1 is not uniruled and the base field is the complex numbers, essentially the same conclusion was proved by Jelonek [12, Theorem 3.7]. For our application, it is crucial that we only need to assume that D_1 is not ruled: every quadric over a field is uniruled, but many of them are not ruled.

Proof. (1) Let Y be the normalization of the closure of the graph of f in $X_1 \times X_2$; then we have proper birational morphisms $f_i : Y \to X_i$ with $f = f_2 f_1^{-1}$. The proper transform of D_1 in Y is a divisor birational to D_1. Since D_1 is not ruled, Lemma 2.1 shows that D_1 maps birationally to its image in X_2. Since D_2 is irreducible, this means that D_1 maps birationally to D_2. Thus, f induces an isomorphism between some open subsets $X_i - S_i$ where S_i has codimension at least 1 in D_i, thus codimension at least 2 in X_i.

(2) The birational map f from X_1 to X_2 is an isomorphism in codimension 1, and it maps the divisor D_1 to the divisor D_2. So f induces an isomorphism between the rings $\oplus_{j \geq 0} H^0(X_i, jD_i)$. (The point is that the group of sections of a line bundle on a normal variety remains unchanged upon removal of a subset of codimension at least 2 from the variety [10, Lemma 2.32]. We are assuming that X_i is regular, hence normal, in a neighborhood of D_i.) Since D_i is ample for $i = 1$ and 2, the projective variety Proj of that graded ring is isomorphic to X_1 or X_2, respectively. So f induces an isomorphism from X_1 to X_2. QED

The same proof works if we replace the assumption that the divisors D_i are ample by the assumption that the varieties X_i are Fano, that is, that their anticanonical line bundles are ample. Both conditions will be satisfied in our application, Theorem 2.3.

We draw the following conclusion about automorphism groups of affine quadrics. Define the first Witt index $i_1(q)$ of an anisotropic quadratic form q of dimension at least 2 over a field to be the Witt index (the maximum dimension of an isotropic subspace) of the quadratic form q over the function field of the projective quadric $\{q = 0\}$ [14]. We always have $i_1(q) \geq 1$. If V denotes the vector space on which q

is defined, we write $GO(V)$ for the subgroup of linear automorphisms that preserve q up to scalars and $PGO(V)$ for $GO(V)/k^*$. More generally, for a linear subspace $W \subset V$, $PGO(W \subset V)$ denotes the subgroup of $PGO(V)$ that maps W into itself.

Theorem 2.3 *Let f be a polynomial of degree at most 2 on a vector space W over a field k, and let q be the homogeneous part of degree 2 of f. Assume that the quadratic form q is anisotropic and that the first Witt index $i_1(q)$ is equal to 1. If k has characteristic 2, assume that the projective quadric $\{q = 0\}$ over k is a regular scheme. By introducing an extra variable to homogenize f, we get a quadratic form on a vector space V which restricts to q on a hyperplane $W \subset V$. Then the automorphism group of the affine quadric $\{f = 0\}$ is equal to $PGO(W \subset V)$. Also, the automorphism group of the complement $\mathbf{P}^{n-1} - \{q = 0\}$, where n is the dimension of q, is equal to $PGO(V)$.*

Proof. For an anisotropic quadratic form q with first Witt index 1 over k, Karpenko proved that the associated projective quadric Q is not ruled over k [13, Theorem 6.4]. Karpenko's theorem was extended to smooth quadrics in characteristic 2 by Elman, Karpenko, and Merkurjev [5] and then to all quadrics in characteristic 2 [21].

Both the affine quadric $\{f = 0\}$ and the complement $\mathbf{P}^{n-1} - \{q = 0\}$ are complements of Q in projective varieties X. Also, X is regular in a neighborhood of Q since the Cartier divisor Q is regular. Finally, the divisor Q is ample on X. By Theorem 2.2, every automorphism of the affine varieties $\{f = 0\}$ and $\mathbf{P}^{n-1} - \{q = 0\}$ extends to an automorphism of the compactification. It is then easy to compute the automorphism group precisely. QED

To see what Theorem 2.3 means more concretely, let us discuss which quadratic forms have first Witt index equal to 1. Briefly, this is the "typical" behavior of quadratic forms of any dimension over a sufficiently complicated field. For example, the generic form $t_1x_1^2 + \cdots + t_nx_n^2$ over $k = k_0(t_1, \ldots, t_n)$ has first Witt index 1 [14, Example 5.7]. See section 3 to see the meaning of the first Witt index for low-dimensional forms. Also, over any field, every anisotropic quadratic form whose dimension has the form $2^a + 1$ has first Witt index 1, by Hoffmann [7] (in characteristic not 2) and Hoffmann-Laghribi [9] (in characteristic 2). (Hoffmann's theorem in characteristic not 2 was later reproved using Rost's degree formula [18] and also by studying Steenrod operations on Chow groups of quadrics [5, Corollary 75.8].) In particular, the form $x_1^2 + \cdots + x_r^2$ over the real numbers has first Witt index 1 whenever r is of the form $2^a + 1$; this case was known earlier by Knebusch [15, Proposition 7.9, Example 7.10]. (The form $x_1^2 + \cdots + x_r^2$ over the real numbers with r not of the form $2^a + 1$ has first Witt index greater than 1, as explained at the end of section 3.) We deduce the following conclusion, proved for $n = 2$ in [20, Theorem 6.2].

Corollary 2.4 *The automorphism group of the sphere*

$$S^n_{\mathbf{R}} = \{x_0^2 + \cdots + x_n^2 = 1\},$$

as an affine algebraic variety over the real numbers, is equal to the orthogonal group $O(n + 1)$ whenever n is a power of 2.

It is not known whether the automorphism group of $S^n_{\mathbf{R}}$ is only the orthogonal group when n is not a power of 2.

Corollary 2.4 is vaguely reminiscent of Wood's theorem that when n is a power of 2, any morphism from the affine variety $S^n_{\mathbf{R}}$ to a lower-dimensional sphere is constant [22]. It is interesting to note that Wood's theorem may not be optimal in high dimensions. It is optimal in low dimensions, since the Hopf maps are nonconstant polynomial maps $S^3 \to S^2$, $S^7 \to S^4$, and $S^{15} \to S^8$. The first open case is: is there a nonconstant polynomial map $S^{48} \to S^{47}$? At least there is no such map given by homogeneous quadratic polynomials [23].

3 Which quadrics are ruled?

The heart of the proof of Theorem 2.3 is the fact that, by Karpenko's theorem, an anisotropic quadratic form with first Witt index 1 is not ruled (that is, the corresponding projective quadric is not ruled). We conjecture the converse (Conjecture 3.1). In this section we give some evidence, including proving the conjecture for quadratic forms of dimension at most 9 (Lemma 3.3).

Conjecture 3.1 *Let q be an anisotropic quadratic form over a field k. If the first Witt index of q is greater than 1, then q is ruled over k.*

For the rest of this section, we will assume that the field k is not of characteristic 2. The conjecture in characteristic 2 is discussed in [21]. Note that the problem of ruledness is only interesting for anisotropic quadrics, since a smooth isotropic quadric of dimension at least 1 over a field k is rational over k and hence ruled over k.

Some evidence for Conjecture 3.1 is that a "stabilized" version is true. Namely, if Q is an anisotropic quadric with $i_1(Q)$ greater than 1, let Q' be any subquadric of codimension $r := i_1(Q) - 1$. Then the definition of $i_1(Q)$ implies that Q' becomes isotropic over the function field $k(Q)$; that is, there is a rational map from Q to Q' over k. Since there is also a rational map from Q' to Q (the inclusion), a standard result in the theory of quadratic forms gives that Q is stably birational to Q' [17, Theorem X.4.25]. That is, $Q \times \mathbf{P}^a$ is birational to $Q' \times \mathbf{P}^{r+a}$ for some a. So it seems plausible (an analogue of the "quadratic Zariski problem" in Ohm [19]) that Q should be birational to $Q' \times \mathbf{P}^r$ and hence that Q should be ruled.

Conjecture 3.1 is true for special Pfister neighbors, by Knebusch. We recall the definitions. We use Lam's book [17, Chapter X] as a reference on the theory of quadratic forms, although our notation is slightly different. Two quadratic forms over a field k are similar if one is isomorphic to a nonzero scalar multiple of the other. An n-fold Pfister form $\langle\langle a_1, \ldots, a_n\rangle\rangle$ is the tensor product of the 2-dimensional quadratic forms $\langle 1, -a_i\rangle$ for $a_1, \ldots, a_n$ in k^*. A Pfister neighbor is a form q similar to a subform of an n-fold Pfister form with dim $q > 2^{n-1}$. A special Pfister neighbor is a form q similar to $\alpha \perp c\alpha'$ for some $(n-1)$-fold Pfister form α and some nonzero subform α' of α; clearly q is a neighbor of the n-fold Pfister form $\alpha \perp c\alpha$. One can check that a neighbor of an n-fold Pfister form is special if and only if it contains a scalar multiple of some $(n-1)$-fold Pfister form. Pfister neighbors are among the simplest quadratic forms: in particular, an anisotropic Pfister neighbor of dimension

$2^{n-1}+a$, where $1 \leq a \leq 2^{n-1}$, has first Witt index a, which is the largest possible for forms of that dimension, by Hoffmann [7].

Thus, Conjecture 3.1 predicts that a Pfister neighbor whose dimension is not of the form $2^{n-1}+1$ should be ruled. This is not known in general, but it is true for special Pfister neighbors by Knebusch; more precisely, he showed that the function field of $\alpha \perp c\alpha'$ is a purely transcendental extension of the function field of the "base form" $\alpha \perp \langle c \rangle$ [14, pp. 73–74]. Ahmad-Ohm observed that Knebusch's argument produces other examples of ruled quadrics, as follows [2, after 1.3]. In particular, every form divisible by a binary form is ruled.

Lemma 3.2 *Let P be a Pfister form over a field k, P_1 a nonzero subform of P, and $b_1, \ldots, b_r$ any elements of k^*. Then the projective quadric associated to $b_1P \perp \cdots \perp b_{r-1}P \perp b_rP_1$ is birational to the product of a projective space with the quadric $b_1P \perp \cdots \perp b_{r-1}P \perp \langle b_r \rangle$; in particular, it is ruled if P_1 has dimension at least 2.*

Proof. A Pfister form P is strongly multiplicative: there is a "multiplication" xy on the vector space of P which is a rational function of x and linear in y such that $P(xy) = P(x)P(y)$ [17, Theorem X.2.11]. Given a general point on the first quadric $b_1P(x_1)+\cdots+b_{r-1}P(x_{r-1})+b_rP(x_r) = 0$ (where $x_r \in P_1$), we map it to a point on the second quadric by noting that $b_1P(x_rx_1)+\cdots+b_{r-1}P(x_rx_{r-1})+b_rP(x_r)^2 = 0$. The general fibers of this rational map are linear spaces. QED

Lemma 3.3 *Let q be an anisotropic quadratic form over a field k of characteristic not 2. If the first Witt index of q is greater than 1 and q has dimension at most 9, then q is ruled.*

Proof. Every form q of dimension 3 or 5 has $i_1(q) = 1$, and so there is nothing to check. A form q of dimension 4 with $i_1(q) > 1$ has $i_1(q) = 2$ and is similar to a Pfister form [17, Theorem X.4.14]. So the corresponding quadric surface is ruled. A form q of dimension 6 with $i_1(q) > 1$ has $i_1(q) = 2$ and is divisible by a binary form, by Knebusch [15, Theorem 10.3]. Therefore q is ruled. A form q of dimension 7 with $i_1(q) > 1$ has $i_1(q) = 3$ [8, Theorem 4.1] and is therefore similar to the pure subform $p' = p - \langle 1 \rangle$ of a Pfister form p [14, Theorem 5.8]. Clearly such a Pfister neighbor is special, and so q is ruled. Next, if q is a form of dimension 8 with $i_1(q) > 1$, then either $i_1(q) = 4$ and q is similar to a 3-fold Pfister form, or $i_1(q) = 2$ and q is divisible by a binary form, by Hoffmann [8, Theorem 4.1]. It follows that q is ruled. Finally, every form q of dimension 9 has $i_1(q) = 1$ and so there is nothing to prove. QED

For 10-dimensional forms, Conjecture 3.1 remains open. By Izhboldin, a 10-dimensional form q with $i_1(q) > 1$ is either divisible by a binary form or is a Pfister neighbor [11, proof of Conjecture 0.10]. In the first case, q is ruled, but the second case is harder. There are 10-dimensional Pfister neighbors which are not special, by Ahmad-Ohm [2, 2.8], and it is not clear how to prove that such forms are ruled.

Another example: for any n, the form $x_0^2+\cdots+x_n^2$ is a special Pfister neighbor over any field. That is clear by viewing this form as a subform of a Pfister form $\langle\langle -1, \cdots, -1 \rangle\rangle$. So, for example over the real numbers, the quadric $\{x_0^2+\cdots+x_n^2 = 0\}$ is ruled if and only if n is not a power of 2, in agreement with Conjecture 3.1.

Acknowledgement. Thanks to Jean-Louis Colliot-Thélène and Detlev Hoffmann for their comments.

References

[1] S. Abhyankar. On the valuations centered in a local domain. *Am. J. Math.* **78** (1956), 321–348.

[2] H. Ahmad and J. Ohm. Function fields of Pfister neighbors. *J. Alg.* **178** (1995), 653–664.

[3] R. Baeza. *Quadratic forms over semilocal rings.* Lecture Notes in Math. vol. 655 (Springer, 1978).

[4] A. Borel. *Linear algebraic groups.* (Springer, 1991).

[5] R. Elman, N. Karpenko, and A. Merkurjev. *Algebraic and geometric theory of quadratic forms.* AMS, to appear.

[6] M. Gizatullin and V. Danilov. Automorphisms of affine surfaces. II. *Math. USSR Izv.* **11** (1977), 51–98.

[7] D. Hoffmann. Isotropy of quadratic forms over the function field of a quadric. *Math. Z.* **220** (1995), 461–476.

[8] D. Hoffmann. Splitting patterns and invariants of quadratic forms. *Math. Nachr.* **190** (1998), 149–168.

[9] D. Hoffmann and A. Laghribi. Isotropy of quadratic forms over the function field of a quadric in characteristic 2. *J. Alg.* **295** (2006), 362–386.

[10] S. Iitaka. *Algebraic geometry.* (Springer, 1982).

[11] O. Izhboldin. Fields of u-invariant 9. *Ann. Math.* **154** (2001), 529–587.

[12] Z. Jelonek. Affine smooth varieties with finite group of automorphisms. *Math. Z.* **216** (1994), 575–591.

[13] N. Karpenko. On anisotropy of orthogonal involutions. *J. Ramanujan Math. Soc.* **15** (2000), 1–22.

[14] M. Knebusch. Generic splitting of quadratic forms. I. *Proc. LMS* **33** (1976), 65–93.

[15] M. Knebusch. Generic splitting of quadratic forms. II. *Proc. LMS* **34** (1976), 1–31.

[16] J. Kollár. *Rational curves on algebraic varieties.* (Springer, 1999).

[17] T. Y. Lam. *Introduction to quadratic forms over fields.* (American Mathematical Society, 2005).

[18] A. Merkurjev. Rost's degree formula. Notes of Lens mini-course, June 2001. http: //www.math.ucla.edu/%7Emerkurev/publicat.htm

[19] J. Ohm. The Zariski problem for function fields of quadratic forms. *Proc. Amer. Math. Soc.* **124** (1996), 1679–1685.

[20] B. Totaro. Complexifications of nonnegatively curved manifolds. *J. Eur. Math. Soc.* **5** (2003), 69–94.

[21] B. Totaro. Birational geometry of quadrics in characteristic 2. *J. Alg. Geom.*, to appear.

[22] R. Wood. Polynomial maps from spheres to spheres. *Invent. Math.* **5** (1968), 163–168.

[23] P. Yiu. Quadratic forms between Euclidean spheres. *Manu. Math.* **83** (1994), 171–181.