The resolution property for schemes and stacks

Burt Totaro

Abstract. We prove the equivalence of two fundamental properties of algebraic stacks: being a quotient stack in a strong sense, and the resolution property, which says that every coherent sheaf is a quotient of some vector bundle. Moreover, we prove these properties in the important special case of orbifolds whose associated algebraic space is a scheme. (Mathematics Subject Classification: Primary 14A20, Secondary 14L30.)

1 Introduction

Roughly speaking, an algebraic stack is an object which looks locally like the quotient of an algebraic variety by a group action [26]. Thus it is a fundamental question whether a given stack is globally the quotient of a variety by a group action. We show that a strong version of this property is equivalent to another fundamental property of stacks: having "enough" vector bundles for geometric purposes. The equivalence turns out to be interesting even in the special case where the stack is a scheme. Moreover, we prove the existence of enough vector bundles in the important case of orbifolds whose associated algebraic space is a scheme. In more detail, the two main results of the paper are:

Theorem 1.1 *Let X be a normal noetherian algebraic stack (over $\mathbf{Z}$) whose stabilizer groups at closed points of X are affine. The following are equivalent.*

(1) X has the resolution property: every coherent sheaf on X is a quotient of a vector bundle on X.

(2) X is isomorphic to the quotient stack of some quasi-affine scheme by an action of the group $GL(n)$ for some n.

For X of finite type over a field k, these are also equivalent to:

(3) X is isomorphic to the quotient stack of some affine scheme over k by an action of an affine group scheme of finite type over k.

Theorem 1.2 *Let X be a smooth Deligne-Mumford stack over a field k. Suppose that X has finite stabilizer group and that the stabilizer group is generically trivial. Let B be the Keel-Mori coarse moduli space of X [20]. If the algebraic space B is a scheme with affine diagonal (for example a separated scheme), then the stack X has the resolution property.*

Informally, Theorem 1.2 says that any "orbifold coherent sheaf" on a scheme with quotient singularities admits a resolution by "orbifold vector bundles." For example, Theorem 1.2 implies that the moduli stacks of curves, $\overline{M}_{g,n}$ with $g \geq 3$ (so that the generic stabilizer is trivial), have the resolution property. This was proved earlier by Mumford [27] by showing that these particular stacks admit a Cohen-Macaulay global cover. The more general Theorem 1.2 has often been wished for, even in the special case where B is quasi-projective. It allows the hard-to-verify assumption of a Cohen-Macaulay global cover to be removed from various papers, such as Kawamata's paper on flips and derived categories [19].

In Theorem 1.1, the fact that quotient stacks $W/GL(n)$ with W a noetherian quasi-affine scheme (an open subset of an affine scheme) have the resolution property is an easy special case of Thomason's results on the resolution property [35], listed in Theorem 2.1 below. So the new implication is the opposite one, which says that stacks with the resolution property are quotient stacks of a very special kind. Edidin, Hassett, Kresch, and Vistoli proved a step in the direction of Theorem 1.1: they showed that a stack with quasi-finite stabilizer group which has the resolution property is a quotient stack $W/GL(n)$ with W an algebraic space, that is, a stack with trivial stabilizer group ([8], Theorem 2.7 and Theorem 2.14). Algebraic spaces (and even schemes) do not all have the resolution property, as explained below, and so we need to strengthen the conclusion as in Theorem 1.1 in order to have an equivalence.

Remarks. (1) The restriction to stacks with affine stabilizer groups in Theorem 1.1 seems reasonable. In fact, I would argue that the resolution property is not meaningful for a stack whose stabilizer groups are not affine. The simplest example of such a stack is the classifying stack BE of an elliptic curve E. Because every linear representation of an elliptic curve is trivial, coherent sheaves and vector bundles on BE are both simply vector spaces. Thus, the resolution property for BE holds, but this has no real geometric significance. In particular, the K-groups of BE defined using either vector bundles or perfect complexes (cf. section 2) are simply the K-groups of a point.

(2) Theorem 1.1 cannot be strengthened to say that a stack X with the resolution property (and affine stabilizer groups) is a quotient stack $W/GL(n)$ with W an affine scheme. Indeed, quotient stacks of the latter form are very special. For example, by geometric invariant theory, any algebraic space which is the quotient of an affine scheme by a reductive group such as $GL(n)$ (as a stack, which means that the group action is free) is in fact an affine scheme ([28], Amplification 1.3).

Theorem 1.1 implies:

Proposition 1.3 *Let X be a noetherian algebraic stack whose stabilizer groups at closed points of X are affine. If X has the resolution property, then the diagonal morphism $X \to X \times_{\mathbf{Z}} X$ is affine.*

For example, a scheme X has affine diagonal morphism if and only if the intersection of any two affine open subsets is affine. The property of having affine diagonal is a natural weakening of separatedness: the smooth non-separated scheme $A^n \cup_{A^n-0} A^n$ has affine diagonal for $n = 1$ but not for $n \geq 2$. Thus Proposition 1.3 immediately implies Thomason's observation that the scheme $A^n \cup_{A^n-0} A^n$ does not have the resolution property for $n \geq 2$ ([37], Exercise 8.6). Most other known counterexamples to the resolution property have non-affine diagonal and so are explained by Proposition 1.3. Unfortunately, not all stacks with affine diagonal have the resolution property. By Grothendieck, there is a normal (affine or projective) complex surface Y which has a non-torsion element of $H^2_{\text{et}}(Y, G_m)$ ([13], II.1.11(b)).

As Edidin, Hassett, Kresch, and Vistoli observed, the corresponding G_m-gerbe X over Y (a stack with stabilizer group G_m at every point) is a stack with affine diagonal which is not a quotient of an algebraic space by $GL(n)$ ([8], Example 3.12). Therefore, Theorem 1.1 says that X fails to have the resolution property. Despite this counterexample, we can ask the following optimistic questions about the resolution property:

Question 1. Let X be a noetherian stack with quasi-finite stabilizer group (for example, an algebraic space or a scheme). Suppose that X has affine diagonal (for example, X separated). Does X have the resolution property?

This would be very useful. I do not know how likely it is. At the moment, the resolution property is not known even for such concrete objects as normal toric varieties or smooth separated algebraic spaces over a field.

Question 2. Let X be a smooth stack over a field such that X has affine diagonal. Does X have the resolution property?

Question 3. Let X be a stack which has the resolution property. Suppose that X has an action of a flat affine group scheme G of finite type over a field or over $\mathbf{Z}$. Does X/G have the resolution property?

The converse to Question 3 is true and sometimes useful: if G acts on a stack X (for example, a scheme) such that the quotient stack X/G has the resolution property, then so does X (Corollary 5.2).

Thomason proved several cases of Question 3, listed in Theorem 2.1 below. One case where his methods do not immediately apply is the quotient stack Q of the nodal cubic curve by the multiplicative group, which has several pathological properties described in section 9. Nonetheless, in that section we use some ideas from the proof of Theorem 1.1 to prove the resolution property for Q. In general, the proof of Theorem 1.1 often shows how to construct a coherent sheaf C on a given stack X such that X has the resolution property if and only if the single sheaf C is a quotient of a vector bundle. I expect that this method should help to prove the resolution property in other situations as well.

Finally, section 8 considers the question of whether surjectivity of the natural map from the Grothendieck group $K_0^{\text{naive}}X$ of vector bundles on X to the group K_0X of perfect complexes is enough to imply the resolution property. The answer is yes for smooth schemes, but not for smooth algebraic spaces or Deligne-Mumford stacks, as we will show in two examples.

I would like to thank Yujiro Kawamata and Gabriele Vezzosi for useful discussions on Theorem 1.2. In particular, the proof given here of Theorem 1.2 has been simplified by an idea of Vezzosi's.

Contents

2 History of the resolution property

One important reason to consider the resolution property is its role in K-theory. For any scheme X, it is natural to consider the Grothendieck group of vector bundles $K_0^{\text{naive}}X$. Quillen extended this group to a sequence of groups $K_*^{\text{naive}}X$ built from the category of vector bundles on X. We use the name "naive" because these groups do not satisfy the important properties such as Mayer-Vietoris for open coverings of arbitrary schemes. Thomason showed how to define the right K-groups K_*X for arbitrary schemes [37]; more generally, it seems that the same definition works for a stack X. Instead of vector bundles, he used perfect complexes, that is, complexes of sheaves which are locally quasi-isomorphic to bounded complexes of finite-dimensional vector bundles. The same idea comes up in topology. In simple situations such as compact Hausdorff spaces, one can define topological K-groups using only finite-dimensional vector bundles. In more complex situations, in order to get the right K-groups (satisfying Mayer-Vietoris), one has to consider Fredholm complexes, that is, complexes of infinite-dimensional vector bundles with finite-dimensional cohomology sheaves. An example where infinite-dimensional bundles are needed is the twisted K-theory recently considered by Freed, Hopkins, Teleman, and others, which can be viewed as the K-theory of a "topological stack," the S^1-gerbe X over a space Y associated to an element of $H^2(Y, S^1_{\text{cont}}) = H^3(Y, \mathbf{Z})$; when this element is not torsion, there are not enough finite-dimensional vector bundles on the stack X ([3], [9]).

It is natural to ask for criteria to ensure that the natural map from $K_*^{\text{naive}}X$ to K_*X is an isomorphism, because this means that Thomason's K-groups give information about vector bundles on X, which are geometrically appealing. Thomason gave a satisfactory answer: if X has the resolution property, then the map from $K_*^{\text{naive}}X$ to K_*X is an isomorphism.

For clarity, perhaps I should add that the G-theory (or K'-theory) of coherent sheaves, also defined by Quillen, has good properties in general and has not had to be modified. In particular, for any regular scheme X, the natural map $K_*X \to G_*X$ is an isomorphism, whereas these groups may differ from $K_*^{\text{naive}}X$ when the resolution property fails, for example for the regular scheme $X = A^n \cup_{A^n - 0} A^n$ with $n \geq 2$ ([37], Exercise 8.6).

The resolution property is known to hold for a vast class of schemes and stacks. In particular, it holds for any noetherian scheme with an ample family of line bundles, by Kleiman and independently Illusie. Kleiman's proof is given in Borelli [5], 3.3, and Illusie's is in [17], 2.2.3 and 2.2.4. A convenient summary of the results on ample families of line bundles is given in Thomason-Trobaugh [37], 2.1. By definition, a scheme X has an ample family of line bundles if X is the union of open affine subsets of the form $\{f \neq 0\}$ with f a section of a line bundle on X. Borelli and Illusie also showed that every regular (or, more generally, factorial) separated noetherian scheme has an ample family of line bundles ([4], 4.2 and [17], 2.2.7). (In fact, the same proof works a little more generally, as observed recently by Brenner and Schröer: every **Q**-factorial noetherian scheme with affine diagonal has an ample family of line bundles ([6], 1.3).) Also, an ample family of line bundles passes to locally closed subschemes, and so the resolution property holds for all quasi-projective schemes over an affine scheme. Finally, Thomason proved the resolution property for the most naturally occurring stacks, quotient stacks of the following types among others ([35], 2.4, 2.6, 2.10, 2.14):

Theorem 2.1 *Let X be a noetherian scheme over a regular noetherian ring R of dimension at most 1. Let G be a flat affine group scheme of finite type over R together with an action of G on X.*

(1) If X has an ample family of G-equivariant line bundles, then the stack X/G has the resolution property.

(2) Suppose that G is an extension of a finite flat group scheme by a smooth group scheme with connected fibers over R. (This is automatic if R is a field.) If X is normal and has an ample family of line bundles, then the stack X/G has the resolution property.

Equivalently, under these assumptions, every G-equivariant coherent sheaf on X is a quotient of a G-equivariant vector bundle on X. For example, it follows that the resolution property holds for one of Hironaka's examples of a smooth proper algebraic space X of dimension 3 which is not a scheme: X is the quotient of a smooth separated scheme (hence a normal scheme with an ample family of line bundles) by a free action of the group $\mathbf{Z}/2$ ([22], pp. 15–17). Hironaka's paper [16], Example 2, gives closely related examples for which the resolution property seems to be unknown.

Hausen showed recently that a scheme (assumed to be reduced and of finite type over an algebraically closed field, although that should be unnecessary) has an ample family of line bundles if and only if it is a quotient $W/(G_m)^n$ for some free action of the torus $(G_m)^n$ on a quasi-affine scheme W ([15], 1.1). This is a satisfying analogue to Theorem 1.1, though the situation is simpler in that only schemes are involved.

We can ask whether the resolution property holds in cases not covered by the above results, but here progress has been slow. As mentioned in the introduction, the resolution property is still an open problem even for such concrete objects as normal toric varieties or smooth separated algebraic spaces. For example, Fulton found a 3-dimensional normal proper toric variety X over $\mathbf{C}$ which has zero Picard group ([10], p. 65); such a variety cannot have an ample family of line bundles, and it is unknown whether the resolution property holds. One encouraging result is the recent proof by Schröer and Vezzosi of the resolution property for all normal separated surfaces ([32], 2.1). There is a normal proper surface over $\mathbf{C}$ that has zero Picard group. Thus, Schröer and Vezzosi were able to construct enough vector bundles to prove the resolution property on such a variety, even though it does not have an ample family of line bundles.

As indicated in the introduction, all known examples of stacks without the resolution property are non-separated. The situation is very different in the complex analytic category, where Voisin has recently proved that the resolution property can fail even for compact Kähler manifolds of dimension 3 ([39], Appendix).

3 Proof of Theorem 1.1, first implications

We begin by proving the easier parts of Theorem 1.1: that (2) implies (1) and that (2) and (3) are equivalent.

First, (2) implies (1). Let X be the quotient stack $W/GL(n)$ for some action of the group $GL(n)$ (over $\mathbf{Z}$) on a noetherian quasi-affine scheme W. A noetherian scheme W is quasi-affine if and only if the trivial line bundle O_W is ample ([12], Proposition II.5.1.2). Given an action of $GL(n)$ on W, the trivial line bundle O_W has the structure of a $GL(n)$-equivariant line bundle in a natural way, so O_W is a $GL(n)$-equivariant ample line bundle on W. By Theorem 2.1, due to Thomason, it follows that every $GL(n)$-equivariant coherent sheaf on W is a quotient of some $GL(n)$-equivariant vector bundle. Equivalently, the stack $W/GL(n)$ has the resolution property.

Next, let us show that (2) implies (3) in Theorem 1.1. Here we only consider stacks of finite type over a field k. The proof is an equivariant version of Jouanolou's trick, which we will prove directly ([18], 1.5). Let X be a quotient stack $W/GL(n)$ for some action of $GL(n)$ over k on a quasi-affine scheme W of finite type over k. By EGA II.5.1.9 [12], W embeds as an open subset of an affine scheme Y of finite type over k; we can also arrange for this embedding to be $GL(n)$-equivariant, simply by taking Spec of a bigger finitely generated subalgebra of $O(W)$. Here we are using that the $GL(n)$-module $O(W)$ over k is a union of finite-dimensional representations of $GL(n)$; a direct proof is given in GIT [28], p. 25, although it is also a special case of the fact that a quasi-coherent sheaf on a noetherian stack (here $BGL(n)$ over k) is a filtered direct limit of its coherent subsheaves ([26], 15.4). Let the closed subset $Y - W$ be defined by the vanishing of regular functions $f_1, \ldots, f_r$ on the affine scheme Y; we can assume that the linear span of the functions $f_1, \ldots, f_r$ is preserved by the action of $GL(n)$, defining a representation $GL(n) \to GL(r)$ over k. Then we have a $GL(n)$-equivariant morphism $\alpha : W \to A^r_k - 0$ defined by

$$w \mapsto (f_1(w), \ldots, f_r(w)).$$

The subsets $\{f_i \neq 0\}$ of W are affine, and so α is an affine morphism.

Define the affine group Aff_{r-1} as the semidirect product $(G_a)^{r-1} \rtimes GL(r-1)$. Then we can identify $A^r - 0$ with the homogeneous space $GL(r)/\mathrm{Aff}_{r-1}$. Define A as the pullback scheme:

$$\begin{array}{ccc} A & \longrightarrow & GL(r) \\ \downarrow & & \downarrow \\ W & \longrightarrow & A^r - 0 \end{array}$$

Since W is affine over $A^r - 0$, A is affine over $GL(r)$, and hence A is an affine scheme. The group $GL(n) \times \mathrm{Aff}_{r-1}$ acts on W and $A^r - 0$, with Aff_{r-1} acting trivially, and it acts on $GL(r)$ via left multiplication by $GL(n)$ (using the representation $GL(n) \to GL(r)$) and right multiplication by Aff_{r-1}. It follows that the pullback scheme A also has an action of $GL(n) \times \mathrm{Aff}_{r-1}$. We see from the diagram that we have an isomorphism of quotient stacks:

$$A/(GL(n) \times \mathrm{Aff}_{r-1}) \cong W/GL(n).$$

So the given stack $X = W/GL(n)$ is the quotient of the affine scheme A by the affine group scheme $GL(n) \times \mathrm{Aff}_{r-1}$. Thus (2) implies (3) in Theorem 1.1.

To prove that (3) implies (2), we need the following well-known fact.

Lemma 3.1 *Every affine group scheme G of finite type over a field k has a faithful representation $G \to GL(n)$ such that $GL(n)/G$ is a quasi-affine scheme.*

Proof. To begin, let $G \to GL(n)$ be any faithful representation of G. By Chow, the homogeneous space $GL(n)/G$ is always a quasi-projective scheme over k. More precisely, there is a representation V of $GL(n)$ and a k-point x in the projective space $P(V^*)$ of lines in V whose stabilizer is G ([7], p. 483).

Let the group $GL(n) \times G_m$ act on V by the given representation of $GL(n)$ and by the action of G_m by scalar multiplication. Then the stabilizer in $GL(n) \times G_m$ of a point $y \in V$ lifting x is isomorphic to G. That is, $(GL(n) \times G_m)/G$ is a quasi-affine scheme. Using the obvious inclusion $GL(n) \times G_m \to GL(n+1)$, the quotient $GL(n+1)/(GL(n) \times G_m)$ is an affine scheme, the "Stiefel manifold" of $(n+1) \times (n+1)$ matrices which are projections of rank 1. So $GL(n+1)/G$ maps to the affine scheme $GL(n+1)/(GL(n) \times G_m)$ with fibers the quasi-affine scheme $(GL(n) \times G_m)/G$. More precisely, the fiber embeds $(GL(n) \times G_m)$-equivariantly as a subscheme of the representation V, by its construction. It follows that $GL(n+1)/G$ is a quasi-affine scheme. QED

We can now prove that (3) implies (2) in Theorem 1.1. Let X be the quotient stack of an affine scheme A of finite type over a field k by the action of an affine group scheme G of finite type over k. By Lemma 3.1, there is a faithful representation $G \to GL(n)$ with $GL(n)/G$ a quasi-affine scheme. Let W be the $GL(n)$-bundle over X associated to the G-bundle $A \to X$, $W = (A \times GL(n))/G$. Then W is an A-bundle over the quasi-affine scheme $GL(n)/G$. Since A is affine, it follows that W is a quasi-affine scheme.

4 From a stack to an algebraic space

We now begin the proof of the main part of Theorem 1.1, that (1) implies (2). Thus, let X be a normal noetherian stack whose stabilizer groups at closed points of X are affine. Suppose that X has the resolution property. We will show that X is isomorphic to the quotient stack $W/GL(n)$ for some quasi-affine scheme W with an action of $GL(n)$. (In Theorem 1.1, we only need X to be normal for this part, the proof that (1) implies (2). Normality will be used in section 5.) The following is a first step.

Lemma 4.1 *Let X be a noetherian stack (over $\mathbf{Z}$) whose stabilizer groups at closed points of X are affine. Suppose that X has the resolution property. Then X is isomorphic to the quotient stack of some algebraic space Z_1 over $\mathbf{Z}$ by an action of the group $GL(n_1)$ for some n_1.*

Proof. Equivalently, we have to find a vector bundle E_1 on X such that the total space Z_1 of the corresponding $GL(n_1)$-bundle over X is an algebraic space. As Edidin, Hassett, Kresch, and Vistoli observed, it is equivalent to require that at every geometric point x of X, the action of the stabilizer group G_x of X on the fiber $(E_1)_x$ is faithful ([8], Lemma 2.12).

By definition of a noetherian stack, there is a smooth surjective morphism from a noetherian affine scheme U to X [26]. We can think of the stack X as the quotient U/R of U by the groupoid $R := U \times_X U$. In these terms, the defining properties of a noetherian stack over $\mathbf{Z}$ are that R is a separated algebraic space over $\mathbf{Z}$ and that both projections $R \to U$ are smooth morphisms of finite type ([26], Proposition 4.3.1). Let $G_U \to U$ be the stabilizer group of X, that is, $G_U = R \times_{U \times U} U$. Here G_U is a group in the category of algebraic spaces over U; it is pulled back from the stabilizer group $G := X \times_{X \times X} X$ over X. Since G_U is a closed subspace of R, it is separated of finite type over U.

A point of a stack X is defined in such a way that a point of X is equivalent to an R-orbit in the underlying topological space of the scheme U. Another way to think of a point is as an isomorphism class of substacks $\mathcal{G}$ of X such that $\mathcal{G}$ is a gerbe over some field k ([26], Corollary 11.4); explicitly, $\mathcal{G}$ is the quotient of the corresponding R-orbit by the restriction of the groupoid R. To say that $\mathcal{G}$ is a gerbe means that there is a field extension F over k such that $\mathcal{G} \times_k F$ is isomorphic to the classifying stack of some group over F. The set $|X|$ of points of X is given the quotient topology from U; in particular, a closed point of X can be identified with a closed R-orbit in U ([26], Corollary 5.6.1).

For any vector bundle E on the stack X, the kernel of the G-action on E is a closed subgroup $H \subset G$ over X, which pulls back to a closed subgroup $H_U \subset G_U$ over U. Given a finite sequence of vector bundles $E_1, \ldots, E_n$ on X with kernel subgroups $H_1, \ldots, H_n \subset G$, the kernel subgroup of the direct sum $E_1 \oplus \cdots \oplus E_n$ is the intersection $H_1 \cap \cdots \cap H_n$. In this way, we can repeatedly cut down the kernel subgroup by finding one vector bundle on X after another, and Lemma 4.1 is proved if this subgroup eventually becomes the trivial group over X.

Lemma 4.2 *Let X be a noetherian stack (over $\mathbf{Z}$) which satisfies the resolution property. Let x be a point of X such that the stabilizer group G of X is affine at x. Then there is a vector bundle E on X whose kernel subgroup is trivial at x.*

Proof. As explained above, x corresponds to a substack $\mathcal{G}$ of X which is a gerbe over some field k. Since X is locally noetherian, there is a finite extension F of k such that $\mathcal{G} \times_k F$ is isomorphic to the classifying stack of a group G_s over F, by [26], 11.2.1 and Theorem 11.3. The assumption means that G_s is affine over F. Moreover, G_s is of finite type over F, since $G \to X$ is of finite type. Therefore G_s has a faithful representation over F. We can view such a representation as a vector bundle on the gerbe $\mathcal{G} \times_k F$. Its direct image to $\mathcal{G}$ is a vector bundle C_0 on $\mathcal{G}$ whose pullback to $\mathcal{G} \times_k F$ is a faithful representation of the group G_s. So the kernel subgroup of C_0 over $\mathcal{G}$ is trivial.

Let $i : \mathcal{G} \to X$ denote the inclusion. The direct image $i_* C_0$ is a quasi-coherent sheaf on X, and therefore a direct limit of coherent sheaves on X ([26], Proposition 13.2.6 and Proposition 15.4). Since X has the resolution property, each of these coherent sheaves is a quotient of a vector bundle on X. Since $C_0 = i^* i_* C_0$ and C_0

is coherent, one of these vector bundles E on X must restrict to a vector bundle on $\mathcal{G}$ which maps onto C_0. It follows that the kernel subgroup of E over $\mathcal{G}$ is trivial. QED

We return to the proof of Lemma 4.1. We are given that the stabilizer group of X at each closed point of X is affine. By Lemma 4.2, it follows that for each closed point x of X, there is a vector bundle E on X whose kernel subgroup $H \to X$ is trivial at x. This does not imply that the kernel subgroup of E is trivial in a neighborhood of E. Nonetheless, since the morphism $H \to X$ has finite type, the dimensions of fibers make sense and are upper semicontinuous. (Indeed, it suffices to check this for the pulled-back group $H_U \to U$, and to consider an etale covering of the algebraic space H_U by a scheme; then we can refer to EGA IV.13.1.3 [12].) Therefore the group $H \to X$ is quasi-finite (that is, of finite type and with finite fibers) over some neighborhood of the point x. The space $|X|$ of points of X is a "sober" noetherian topological space (every irreducible closed subset of $|X|$ has a unique generic point) by [26], Corollary 5.7.2. It follows that every open subset of $|X|$ which contains all the closed points must be the whole space. Since $|X|$ is also quasi-compact, there are finitely many vector bundles $E_1, \ldots, E_n$ on X such that the direct sum $E_1 \oplus \cdots \oplus E_n$ has kernel group which is quasi-finite over all of X. Let E now denote this direct sum. In particular, the kernel subgroup $H \to X$ of E is affine over every point of X.

Since the kernel group $H_U \to U$ is quasi-finite, it is finite over a dense open subset V of U ([14], exercise II.3.7). By Lemma 4.2, for every point $x \in V$, there is a vector bundle F on X whose kernel subgroup is trivial at x. Since $H_{E\oplus F}$ is a closed subgroup scheme of H_E, it is finite over V, while also being trivial at x. Therefore $H_{E\oplus F}$ is trivial over some neighborhood of x. By quasi-compactness of V, there is a vector bundle on X (again to be called E) whose kernel subgroup is quasi-finite over U and trivial over V. Then this kernel subgroup will be finite over a larger open subset of U, containing a dense open subset of $U - V$, and so we can repeat the process. By noetherian induction, we end up with a vector bundle on $X = U/R$ with trivial kernel subgroup over all of U. QED

5 From an algebraic space to a scheme

In this section, we will complete the proof of Theorem 1.1.

Let X be a normal noetherian stack X with affine stabilizer groups at closed points of X. Suppose that X satisfies the resolution property. We will show that $X = W/GL(n)$ for some quasi-affine scheme W and some n. By Lemma 4.1, we know that there is a $GL(n_1)$-bundle Z_1 over X for some n_1 which is at least an algebraic space. Since X is normal, Z_1 is normal. We use the name E_1 for the vector bundle on X that corresponds to the $GL(n_1)$-bundle Z_1.

By Artin, since Z_1 is a normal noetherian algebraic space, it is the coarse geometric quotient of some normal scheme A by the action of a finite group G ([23], 2.8; [26], 16.6.2). Moreover, the morphism $\pi : A \to Z_1$ is finite. (In general, the quotient morphism even of an affine noetherian scheme by a finite group need not be a finite morphism, by Nagata [29], but the situation here is better because we know Z_1 is noetherian to start with.) Let $\pi : A \to Z_1$ be the corresponding morphism. Let $U_1, \ldots, U_r$ be an open affine covering of the scheme A. Let S_i be the closed subset $A - U_i$, which we give the reduced subscheme structure. Let I_{S_i} be the corresponding ideal sheaf (the kernel of $O_A \to O_{S_i}$), and let C be the coherent sheaf $C = \oplus_{i=1}^r I_{S_i}$ on A. Then $D := \pi_* C$ is a coherent sheaf on Z_1 because $\pi : A \to Z$ is proper.

In order to use the resolution property for X again, we need a suitable coherent sheaf on $X = Z_1/GL(n)$. That will be supplied by the following lemma.

Lemma 5.1 *Let Z_1 be a noetherian algebraic space (over $\mathbf{Z}$). Let G be a flat affine group scheme over $\mathbf{Z}$ or over a field which acts on Z_1. Then any coherent sheaf on Z_1 is a quotient of some G-equivariant coherent sheaf on Z_1.*

Proof. The morphism α from Z_1 to the quotient stack Z_1/G is affine. So, for every coherent sheaf D on Z_1, the natural map

$$\alpha^* \alpha_* D \to D$$

is surjective. Here $\alpha_* D$ is a quasi-coherent sheaf on Z_1/G. Thus, we have exhibited the coherent sheaf D as the quotient of a G-equivariant quasi-coherent sheaf on Z_1. By Laumon and Moret-Bailly, every quasi-coherent sheaf on a noetherian stack is the filtered direct limit of its coherent subsheaves ([26], Proposition 15.4). So D is in fact the quotient of some G-equivariant coherent sheaf on Z_1. QED

Before continuing with the proof of Theorem 1.1, note the following corollary which could be useful in checking the resolution property in examples. For example, to prove the resolution property for all coherent sheaves on a toric variety, it suffices to prove it for the equivariant coherent sheaves. Klyachko's algebraic description of the equivariant vector bundles on a toric variety should be useful for the latter problem [21].

Corollary 5.2 *Let Z_1 be a noetherian algebraic space (over $\mathbf{Z}$). Let G be a flat affine group scheme of finite type over $\mathbf{Z}$ or over a field which acts on Z_1. If the resolution property holds for the stack Z_1/G, then it holds for Z_1.*

Proof of Corollary 5.2. Every coherent sheaf on Z_1 is a quotient of a G-equivariant coherent sheaf by Lemma 5.1, which in turn is a quotient of a G-equivariant vector bundle on Z_1 by the resolution property for Z_1/G. QED

We now return to the proof of Theorem 1.1. We apply Lemma 5.1 to the algebraic space Z_1 with $X = Z_1/GL(n_1)$ and the coherent sheaf $D = \pi_* C$ on Z_1 defined above. It follows that $\pi_* C$ is the quotient of some $GL(n_1)$-equivariant coherent sheaf D_1 on Z_1. Since the stack X has the resolution property, there is a vector bundle E_2 on X which maps onto D_1, viewed as a coherent sheaf on X. Thus, writing α for the $GL(n_1)$-bundle $Z_1 \to X$, we have surjections

$$\alpha^* E_2 \twoheadrightarrow D_1 \twoheadrightarrow \pi_* C$$

on Z. Since the morphism $\pi : A \to Z_1$ is finite, it is affine. So the natural map $\pi^* \pi_* C \to C$ of sheaves over A is surjective. Thus, we have found a vector bundle

E_2 on X whose pullback $\pi^*\alpha^*E_2$ to the scheme A maps onto the coherent sheaf $C = \oplus_{i=1}^n I_{S_i}$.

Let Z_2 be the $GL(n_2)$-bundle over X associated to the vector bundle E_2. Define W and Y as the indicated pullbacks:

$$\begin{array}{ccc} W & \xrightarrow[GL(n_2)]{\beta} & A \\ \downarrow & & \downarrow \pi \\ Y & \xrightarrow[GL(n_2)]{} & Z_1 \\ \downarrow & & \alpha \downarrow GL(n_1) \\ Z_2 & \xrightarrow[GL(n_2)]{} & X \end{array}$$

Thus W is the $GL(n_2)$-bundle over the scheme A associated to the vector bundle $\pi^*\alpha^*E_2$, which we know maps onto the coherent sheaf $C = \oplus_{i=1}^n I_{S_i}$. It follows that the scheme W is quasi-affine, by the following argument. By construction of W, the vector bundle $\pi^*\alpha^*E_2$ on A pulls back to the trivial bundle on W. Let $\beta : W \to A$ denote this $GL(n_2)$-bundle. Then the coherent sheaf $\beta^*(\oplus_{i=1}^n I_{S_i}) = \oplus_{i=1}^n I_{\beta^{-1}(S_i)}$ on W is spanned by its global sections. This means that for each $1 \leq i \leq n$, the open subset $W - \beta^{-1}(S_i)$ is the union of its open subsets of the form $\{f_{ij} \neq 0\}$, for certain regular functions $f_{i1}, \ldots , f_{ir}$ on W which vanish on $\beta^{-1}(S_i)$. Moreover, each subset S_i was chosen so that $A - S_i$ is an affine scheme. Since $\beta : W \to A$ is an affine morphism, $W - \beta^{-1}(S_i)$ is also an affine scheme. It follows that the open subsets $\{f_{ij} \neq 0\}$ of $W - \beta^{-1}(S_i)$ are affine. Thus, using all i and j, W is the union of open affine subschemes of the form $\{f \neq 0\}$ for regular functions f on W. This means that the scheme W is quasi-affine ([12], Proposition II.5.1.2).

By the pullback diagram above, Y is a $GL(n_1) \times GL(n_2)$-bundle over X. Because Y is a $GL(n_2)$-bundle over the algebraic space Z_1, Y is an algebraic space. Moreover, as the pullback diagram shows, we have a finite surjective morphism from the quasi-affine scheme W to Y. In general, this does not imply that Y is a quasi-affine scheme, as Grothendieck observed ([12], Remark II.6.6.13); for example, there is a non-quasi-affine scheme whose normalization is quasi-affine.

We know, however, that Y is normal, and that Y is the coarse geometric quotient of the quasi-affine scheme W by a finite group G. It follows by the usual construction of quotients by finite group actions that Y is a quasi-affine scheme. Namely, since W is quasi-affine, every finite subset of W is contained in an affine open subset of the form $\{f \neq 0\}$ for some regular function f on W. In particular, each G-orbit in W is contained in an affine open subset. This subset can be taken to be of the form $\{f \neq 0\}$ for some G-invariant function f, by taking the product of the translates of a given function on W. Then we can define the geometric quotient Y of W by G as a scheme, the union of open subsets Spec $O(U)^G$ corresponding to these affine open subsets U of W; a reference that works in this generality is SGA 3 ([11], Theorem 4.1). Finally, the scheme Y thus defined is quasi-affine because it is an open subset of the affine scheme Spec $O(W)^G$. (The rings $O(W)$ and $O(W)^G$ may not be noetherian, but that does not matter for the purpose of proving that Y is quasi-affine.)

We can then consider the $GL(n_1+n_2)$-bundle over X associated to the vector bundle $E_1 \oplus E_2$. Its total space is a bundle over Y with fiber the affine scheme $GL(n_1+n_2)/(GL(n_1)\times GL(n_2))$ (a Stiefel manifold as in section 3), and hence is a quasi-affine scheme. Theorem 1.1 is proved. QED

6 Proof of Proposition 1.3

We now prove Proposition 1.3. That is, let X be a noetherian stack (over $\mathbf{Z}$) with affine stabilizer groups at closed points of X. Suppose that X has the resolution property. We will show that the diagonal morphism $X \to X \times_{\mathbf{Z}} X$ is affine. In this section, I will write $X \times Y$ to mean $X \times_{\mathbf{Z}} Y$.

Suppose first that X is normal. By Theorem 1.1, X is isomorphic to the quotient of some quasi-affine scheme W by an action of $GL(n)$. A quasi-affine scheme W is separated (that is, the diagonal morphism $W \to W \times W$ is a closed embedding). In particular, W has affine diagonal. Since $GL(n)$ is an affine group scheme over $\mathbf{Z}$, it follows that $X = W/GL(n)$ has affine diagonal. This is an easy formal argument, as follows. For brevity, let us write G for $GL(n)$. Since $W \to W/G$ is faithfully flat, to show that $W/G \to W/G \times W/G$ is affine is equivalent to showing that the pulled-back map over $W \times W$ is affine, that is, that $G \times W \to W \times W$, $(g,w) \mapsto (w, gw)$, is affine. But that map is the composition of the map $G \times W \to G \times W \times W$ by $(g,w) \mapsto (g,w,gw)$, which is a pullback of the diagonal map of W and hence is affine, with the projection map $G \times W \times W \to W \times W$ which is affine since G is affine.

For an arbitrary noetherian stack X with affine stabilizer groups at closed points, the proof of Theorem 1.1 works until the last step. We find that X is isomorphic to $Y/GL(n)$ for some noetherian algebraic space Y which admits a finite surjective morphism $W \to Y$ from a quasi-affine scheme W. The point now is that Y is separated since W is, by the following lemma. In particular, Y has affine diagonal, and so the stack $X = Y/GL(n)$ has affine diagonal. Proposition 1.3 is proved. QED

Lemma 6.1 *Let $f : X \to Y$ be a proper surjective morphism of noetherian algebraic spaces. If X is separated, then Y is separated.*

Proof. We have the commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X \times X \\ {\scriptstyle f}\downarrow & & \downarrow \\ Y & \xrightarrow{g} & Y \times Y \end{array}$$

The map $X \to X \times X$ is proper since X is separated, and $X \times X \to Y \times Y$ is proper, so the composition $X \to Y \times Y$ is proper. By the defining properties of a noetherian stack over $\mathbf{Z}$ (as in section 4 or [26]), the diagonal morphism $g : Y \to Y \times Y$ is separated and of finite type. We would like to conclude that $g : Y \to Y \times Y$ is proper, that is, Y is separated, by EGA II.5.4.3 [12]: "Let $f : A \to B$, $g : B \to C$ be morphisms of schemes such that $g \circ f$ is proper. If g is separated of finite type and

f is surjective, then g is proper." We have algebraic spaces rather than schemes here, but the same proof works, as follows. By definition of properness for a map of algebraic spaces ([22], Definition II.7.1), since we know that g is separated of finite type, we only have to show that g is universally closed. The hypotheses pull back under arbitrary morphisms of algebraic spaces, so it suffices to show that the morphism g is closed, that is, that the images of closed sets are closed. But this is clear from surjectivity of f together with the fact that the morphism $g \circ f$ is closed. QED

7 Varieties with quotient singularities: proof of Theorem 1.2

We now prove Theorem 1.2. Thus, let X be a smooth Deligne-Mumford stack over a field k. Suppose that X has finite stabilizer group and that the stabilizer group is generically trivial. Let B be the Keel-Mori coarse moduli space of X [20]. Finally, suppose that the algebraic space B is a scheme with affine diagonal (for example a separated scheme). We will show that the stack X has the resolution property.

We need the following important property of the Keel-Mori space, which I have stated in the full generality in which that space is defined.

Lemma 7.1 *Let X be a stack of finite type over a locally noetherian base scheme S. Suppose that X has finite stabilizer group, so that there is a Keel-Mori quotient space B. Then the map $X \to B$ is proper.*

This is a more general version of Keel and Mori [20], 6.4, which in turn is modeled on Kollár [25], 2.9.

Proof. Here properness for a map of stacks is defined in [26], Chapter 7. In the case at hand, there is a finite surjective morphism from a scheme to X ([8], Theorem 2.7). As a result, there is a valuative criterion for properness using only discrete valuation rings ([26], Proposition 7.12), in which we ask for a lift after a suitable ramified extension of DVRs.

The problem is local over S, so we can assume that S is noetherian. So X is noetherian. Therefore we can find a smooth surjective morphism from an affine scheme U to X. As in section 4, X is the quotient stack of U by the groupoid $R := U \times_X U$. By property 1.8US of the Keel-Mori quotient, the map $U \to B$ is a universal submersion (as defined in EGA IV.15.7.8 [12]). Therefore, as Kollár explains, if T is the spectrum of a DVR and $u : T \to B$ is a morphism then there is a dominant morphism, which we can assume to be finite, from another DVR T' to T, such that the composition $T' \to T \to B$ lifts to a map $\overline{u} : T' \to U$. In the valuative criterion for properness of $X \to B$, we are also given a lift v of u restricted to the general point t_g of T, $v : t_g \to U$. By property 1.8G of the Keel-Mori quotient, $U(\xi)/R(\xi) \to B(\xi)$ is a bijection for every geometric point ξ, and so $\overline{u}|_{t'_g}$ and v become equivalent under the groupoid R after base extension to another DVR T'' which is finite over T'. Since $\overline{u}$ is defined on all of T'', this checks the valuative criterion: the morphism $X \to B$ is proper. QED

We return to the proof of Theorem 1.2. Since X is a smooth Deligne-Mumford stack with trivial generic stabilizer, it is the quotient of some algebraic space Z_1 by an action of $GL(n_1)$ over k, by Edidin, Hassett, Kresch, and Vistoli ([8], Theorem 2.18). (In characteristic zero, this is essentially Satake's classical observation that an orbifold with trivial generic stabilizer is a quotient of a manifold by a compact Lie group, using the frame bundle corresponding to the tangent bundle ([31], p. 475).)

By Laumon and Moret-Bailly (generalized by Edidin, Hassett, Kresch, and Vistoli, as used in the above proof), there is a finite surjective morphism from a scheme A to the Deligne-Mumford stack X ([26], Theorem 16.6). Some special cases of this result were known before, by Seshadri [33], 6.1, and Vistoli [38], 2.6. By Lemma 7.1, the morphism $X \to B$ is proper, and so the composition $A \to B$ is proper. It is clearly also a quasi-finite morphism of algebraic spaces, and so it is finite, in particular affine. Define Z_A by the following pullback diagram.

$$\begin{array}{ccc} Z_A & \longrightarrow & Z_1 \\ \big\downarrow & & \big\downarrow {\scriptstyle GL(n_1)} \\ A & \longrightarrow & X \\ & & \big\downarrow \\ & & B \end{array}$$

Since $Z_A \to A$ is a $GL(n_1)$-bundle and $A \to B$ is finite, both morphisms are affine, and so the composition $Z_A \to B$ is affine. Since $Z_A \to Z_1$ is a finite surjective morphism of algebraic spaces over B, it follows from Chevalley's theorem for algebraic spaces ([22], III.4.1) that the morphism $Z_1 \to B$ is affine.

Since B is a scheme and the morphism $Z_1 \to B$ is affine, the smooth algebraic space Z_1 is a scheme. Likewise, since B has affine diagonal and $Z_1 \to B$ is affine, Z_1 has affine diagonal. To see this, write the diagonal morphism $Z_1 \to Z_1 \times_k Z_1$ as the composition of two maps. The first is $Z_1 \to Z_1 \times_B Z_1$, which is a closed embedding and hence affine since $Z_1 \to B$ is affine and hence separated. Next is $Z_1 \times_B Z_1 \to Z_1 \times_k Z_1$, which is a pullback of the affine morphism $B \to B \times_k B$ and hence is affine.

Thus, Z_1 is a smooth scheme with affine diagonal. So Z_1 has an ample family of line bundles, by Brenner and Schröer (or Borelli and Illusie, in the separated case), as mentioned in section 2. Then, by Theorem 2.1, due to Thomason, the quotient stack $X = Z_1/GL(n_1)$ has the resolution property. QED

8 The resolution property and K-theory

As mentioned in section 2, if a stack X has the resolution property, then the natural map $K_0^{\text{naive}} X \to K_0 X$ is an isomorphism. In particular, every perfect complex on X is equivalent in the Grothendieck group $K_0 X$ of perfect complexes to a difference of vector bundles. We will present two examples to show that the converses to these statements are false. First, we state a positive result for smooth schemes.

Proposition 8.1 *Let X be a smooth scheme of finite type over a field. The following are equivalent.*

(1) X has affine diagonal. For a scheme, as here, it is equivalent to say that the intersection of any two affine open subsets of X is affine.

(2) X is a scheme with an ample family of line bundles.

(3) X has the resolution property.

(4) The natural map from $K_0^{naive}X$ to $K_0X = G_0X$ is surjective. Equivalently, every coherent sheaf on X is equivalent in the Grothendieck group G_0X to a difference of vector bundles.

This is fairly easy, but perhaps suggestive. From my point of view, the interesting equivalence here is between (1) and (3), because one would often like to know whether the resolution property holds, and it is usually easy to check whether a scheme or stack has affine diagonal. We can hope that the equivalence between (1) and (3) holds in much greater generality; see Questions 1 and 2 in the introduction. In more general situations, property (2) will imply the resolution property but will definitely not be equivalent to it, as can be seen from several examples in section 2. Finally, the end of this section will present two examples showing that property (4) does not imply the resolution property in more general situations, for example for smooth algebraic spaces.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are discussed in section 2. First, Brenner and Schröer observed that (1) implies (2), that is, that a smooth scheme with affine diagonal has an ample family of line bundles. The proof is the same as in the case of a smooth separated scheme, due to Borelli and Illusie. Next, Kleiman and Illusie proved that (2) implies (3). Finally, Thomason proved that $K_*^{\text{naive}}X \to K_*X$ is an isomorphism when X has the resolution property. The special case that (3) implies (4) is particularly simple: using the resolution property, every coherent sheaf has a resolution by vector bundles, which can be stopped after finitely many steps because the scheme X is regular.

It remains to show that (4) implies (1). That is, if X is a smooth scheme that does not have affine diagonal, we will find a coherent sheaf C on X whose class in $K_0X = G_0X$ is not a difference of vector bundles. The following proof extends an argument by Schröer and Vezzosi ([32], Proposition 4.2).

The assumption that X does not have affine diagonal means that X has open affine subsets U and V such that $U \cap V$ is not affine. As is well known, the complement of an irreducible divisor D in a smooth affine variety U is affine. Indeed, D is Cartier because U is smooth, and so the inclusion from $U - D$ into U is an affine morphism, which implies that $U - D$ is affine since U is. As a result, if $U - (U \cap V)$ contains any irreducible divisor, we can remove it from U without changing the properties we have stated for U, and likewise for V. Thus we can assume that $U - (U \cap V)$ and $V - (U \cap V)$ have codimension at least 2.

Since U and V are smooth, in particular normal, it follows that the restriction maps from $O(U)$ or $O(V)$ to $O(U \cap V)$ are isomorphisms. Since U and V are affine, this means that both U and V are isomorphic to Spec $O(U \cap V)$. Let S be the closure in X of $U - (U \cap V)$, and let T be the closure in X of $V - (U \cap V)$. We give these closed subsets the reduced scheme structure. Suppose that the coherent sheaf O_S on X is a difference of vector bundles $E - F$ in the Grothendieck group G_0X; we will derive a contradiction.

After shrinking U to a smaller affine open neighborhood of the generic point of an irreducible component of S, and shrinking V to the corresponding neighborhood of the generic point of a component of T, we can assume that the vector bundles E and F are trivial on both U and V. So each of these vector bundles is described up to isomorphism on $U \cup V$ by an attaching map $U \cap V \to GL(n)$. Since $U - (U \cap V)$ has codimension at least 2, every such map extends to U. It follows that E and F are in fact trivial on $U \cup V$. Thus, our assumption implies that the class of O_S in G_0X restricts to zero in $G_0(U \cup V)$.

Let $S_U = S \cap U = U - (U \cap V)$ and $T_V = T \cap V = V - (U \cap V)$; these are disjoint nonempty closed subsets of $U \cup V$, and they are isomorphic. Consider the localization sequences in G-theory, due to Quillen [30]:

$$G_1(U \cap V) \longrightarrow G_0 S_U \longrightarrow G_0 U \longrightarrow G_0(U \cap V) \longrightarrow 0$$

$$G_1(U \cap V) \longrightarrow G_0 T_V \longrightarrow G_0 V \longrightarrow G_0(U \cap V) \longrightarrow 0$$

$$G_1(U \cap V) \longrightarrow G_0(S_U \textstyle\coprod T_V) \longrightarrow G_0(U \cup V) \longrightarrow G_0(U \cap V) \longrightarrow 0$$

Since the inclusions of $U \cap V$ into U and into V are isomorphic, any element of $G_1(U \cap V)$ has the same image in $G_0 S_U$ as in $G_0 T_V$, with respect to the isomorphism of S_U with T_V. Furthermore, the class of O_S in $G_0 S_U$ is not zero. So the class of O_S in $G_0(S_U \coprod T_V) = G_0 S_U \oplus G_0 T_V$ is not in the image of $G_1(U \cap V)$. Thus the class of O_S in $G_0(U \cup V)$ is not zero, contradicting the previous paragraph. So in fact the class of the coherent sheaf O_S in G_0X is not a difference of vector bundles. We have proved that (4) implies (1). QED

Example 1. There is a smooth algebraic space Z such that $K_0^{\text{naive}} Z \to K_0 Z$ is surjective, that is, every perfect complex on Z is equivalent in $K_0 Z$ to a difference of vector bundles, but Z does not have the resolution property.

In conformity with Question 1 in the introduction, the space we construct will not have affine diagonal. Let Y_r be the smooth non-separated scheme $Y = A^r \cup_{A^r - 0} A^r$, $r \geq 2$, over some field k of characteristic not 2. The algebraic space Z_r will be the quotient of Y_r by a free action of the group $\mathbf{Z}/2$, acting by -1 on A^r and switching the two origins. The algebraic space Z_r is a well known example, described and illustrated by Artin [1] and named by Kollár a bug-eyed cover [24]. Its best known property is that it is not locally separated at the image of the origin, and therefore not a scheme.

For $r = 1$, the algebraic space Z_1 has the resolution property. Indeed, Y_1 is a smooth scheme with affine diagonal, and so it has an ample family of line bundles by the result of Brenner and Schröer mentioned in section 2. By Theorem 2.1, it follows that $Z_1 = Y_1/(\mathbf{Z}/2)$ has the resolution property. One gets a more direct proof by observing that the scheme $Y_1 = A^1 \cup_{A^1 - 0} A^1$ is the quotient of $A^2 - 0$ by the diagonal torus $G_m \subset SL(2)$. Likewise, the algebraic space Z_1 is the quotient of the quasi-affine scheme $A^2 - 0$ by the normalizer of G_m in $SL(2)$. (This normalizer is a non-split extension of $\mathbf{Z}/2$ by G_m.) Then it is immediate from Theorem 2.1 that Y_1 and Z_1 have the resolution property.

We now consider $r \geq 2$. Since Z_r does not have affine diagonal, we know by Proposition 1.3 that Z_r does not have the resolution property, as we will see more explicitly below. To compute the group $K_0^{\text{naive}} Z_r$, we need to describe the vector bundles on Z_r. Let $\sigma : A^r - 0 \to A^r - 0$ be multiplication by -1. A vector bundle E on Z_r is a vector bundle E on A^r together with an isomorphism

$$f : E \xrightarrow{\cong} \sigma^* E$$

over $A^r - 0$, such that the composition

$$E \xrightarrow{f} \sigma^* E \xrightarrow{\sigma^* f} \sigma^*\sigma^* E = E$$

is the identity on $A^r - 0$. Since $r \geq 2$, f extends uniquely to a map $f : E \to \sigma^* E$ over all of A^r, and the above composition is the identity over A^r because this is true over $A^r - 0$. Thus the category of vector bundles over Z_r is equivalent to that of $\mathbf{Z}/2$-equivariant vector bundles over A^r, with $\mathbf{Z}/2$ acting on A^r by multiplication by -1. So

$$\begin{aligned} K_0^{\text{naive}} Z_r &\cong K_0^{\text{naive}}(A^r/\mathbf{Z}/2) \\ &\cong K_0(A^r/\mathbf{Z}/2) \\ &\cong \text{Rep}(\mathbf{Z}/2) \\ &\cong \mathbf{Z}^2. \end{aligned}$$

Here $A^r/\mathbf{Z}/2$ denotes the quotient stack of A^r by $\mathbf{Z}/2$. Its naive K-theory coincides with its true K-theory because it has the resolution property, by Theorem 2.1. The calculation that $K_0(A^r/\mathbf{Z}/2)$ is isomorphic to the representation ring of $\mathbf{Z}/2$ follows from the homotopy invariance of equivariant algebraic K-theory, also proved by Thomason ([34], 4.1).

Next, we compute the true K-theory $K_0 Z_r$, which is isomorphic to the Grothendieck group $G_0 Z_r$ of coherent sheaves because the algebraic space Z_r is smooth over k. By the previous paragraph, we can identify the map $K_0^{\text{naive}} Z_r \to K_0 Z_r$ with the pull-back map $G_0(A^r/\mathbf{Z}/2) \to G_0 Z_r$ associated to the obvious flat morphism from Z_r to the quotient stack $A^r/\mathbf{Z}/2$. We have exact localization sequences, by Thomason's paper on equivariant K-theory ([34], 2.7):

$$\begin{array}{ccccccc} G_0(\text{point}/\mathbf{Z}/2) & \longrightarrow & G_0(A^r/\mathbf{Z}/2) & \longrightarrow & G_0((A^r - 0)/\mathbf{Z}/2) & \longrightarrow & 0. \\ \downarrow & & \downarrow & & \downarrow & & \\ G_0((2\text{ points})/\mathbf{Z}/2) & \longrightarrow & G_0 Z_r & \longrightarrow & G_0((A^r - 0)/\mathbf{Z}/2) & \longrightarrow & 0. \end{array}$$

The left vertical map sends $\text{Rep}(\mathbf{Z}/2) \cong \mathbf{Z}^2$ to $G_0(\text{point}) = \mathbf{Z}$ by the rank; in particular, it is surjective. The right vertical map is an isomorphism, and so the center vertical map is surjective. This means that $K_0^{\text{naive}} Z_r \to K_0 Z_r$ is surjective, as we want.

We can also compute $K_0 Z_r$ $(= G_0 Z_r)$ explicitly, using the above diagram. By a Koszul resolution, the pushforward map

$$\text{Rep}(\mathbf{Z}/2) = G_0(\text{point}/\mathbf{Z}/2) \to G_0(A^r/\mathbf{Z}/2) = \text{Rep}(\mathbf{Z}/2)$$

is multiplication by $\lambda_{-1} V := \sum (-1)^i \Lambda^i V$, where V denotes the representation of $\mathbf{Z}/2$ on A^r. Therefore $K_0 Z_r$ is the quotient of $\text{Rep}(\mathbf{Z}/2)$ by the relation

$$W \cdot \lambda_{-1} V = (\dim W) \cdot \lambda_{-1} V$$

for all $W \in \text{Rep}(\mathbf{Z}/2)$. We thereby compute that $K_0 Z_r$ is the quotient of $K_0^{\text{naive}} Z_r = \text{Rep}(\mathbf{Z}/2) = \mathbf{Z} \oplus \mathbf{Z}u$, where u is the nontrivial 1-dimensional representation of $\mathbf{Z}/2$, by the relation $2^r(1-u) = 0$, so that $K_0 Z_r$ is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}/2^r$. In particular, we see again that the resolution property fails for Z_r, because the map $K_0^{\text{naive}} Z_r \to K_0 Z_r$ is not an isomorphism. Explicitly, let O_{A^r} and L denote the $\mathbf{Z}/2$-equivariant line bundles on A^r associated to the trivial and the nontrivial 1-dimensional representations of $\mathbf{Z}/2$. Let K be the coherent sheaf

$$K = \ker(O_{A^r} \oplus L \to O_0),$$

where both O_{A^r} and L map onto O_0. Then K is not a $\mathbf{Z}/2$-equivariant coherent sheaf on A^r, but it is $\mathbf{Z}/2$-equivariant outside the origin, and so it corresponds to a coherent sheaf on X_r. The sheaf K is not a quotient of a vector bundle on X_r.

It is amusing to observe that, over the complex numbers, the non-separated scheme $Y_r = A^r \cup_{A^r - 0} A^r$ is weak homotopy equivalent to the sphere S^{2r}, and the quotient algebraic space Z_r is weak homotopy equivalent to real projective space $\mathbf{RP}^{2r}$. The true K-group $K_0 Z_r = \mathbf{Z} \oplus \mathbf{Z}/2^r$ maps isomorphically to the topological K-group $K^0_{\text{top}} \mathbf{RP}^{2r}$, as computed by Atiyah ([2], p. 107). This suggests that the K-theory of any stack with affine stabilizer group should be closely related to its topological or etale K-theory. The relation will not always be visible on the level of K_0, but rather in the groups K_i with i large. Precisely, there should be an isomorphism

$$K_*(X; \mathbf{Z}/l^\nu)[\beta^{-1}] \to K^*_{\text{et}}(X; \mathbf{Z}/l^\nu),$$

where β denotes the Bott map. In fact, one problem here is to define the groups on the right. For X a locally separated algebraic space of finite type over a reasonable base scheme, or more generally for quotient stacks X/G of such an algebraic space by a linear algebraic group, Thomason proved analogous results in G-theory ([36], 3.17 and Theorem 5.9), which coincides with K-theory for regular stacks. Some of his results apply to the above example $Z_r = Y_r/\mathbf{Z}/2$.

Example 2. There is a smooth Deligne-Mumford stack X for which the natural map $K_0^{\text{naive}} X \to K_0 X$ is an isomorphism, but X does not have the resolution property.

As above, let Y_r be the smooth non-separated scheme $A^r \cup_{A^r - 0} A^r$ over a field k of characteristic not 2. Let X_r be the quotient stack of Y_r by the action of $\mathbf{Z}/2$ which is the identity outside the origin and which switches the two origins. We will show that X_r has the desired properties for $r \geq 2$.

For $r = 1$, X_1 does have the resolution property. Indeed, Y_1 is a smooth scheme with affine diagonal and hence has an ample family of line bundles by Brenner and Schröer, as mentioned in section 2. Therefore the quotient stack $X_1 = Y_1/\mathbf{Z}/2$ has the resolution property by Theorem 2.1. More explicitly, the stack X_1 has the resolution property, by Theorem 2.1, because it is the quotient of the quasi-affine scheme $A^2 - 0$ by the orthogonal group $O(2)$. (The orthogonal group $O(2)$ of the quadratic form $x_1 x_2$ is a split extension of $\mathbf{Z}/2$ by G_m.)

For $r \geq 2$, one checks (by arguments as in Example 1) that pulling back via the flat morphism $X_r \to A^r/\mathbf{Z}/2$ induces an equivalence of categories of vector bundles. Here $A^r/\mathbf{Z}/2$ denotes the quotient stack of A^r by the trivial action of $\mathbf{Z}/2$, and so a vector bundle on $A^r/\mathbf{Z}/2$ is simply a direct sum of bundles $E_0 \oplus E_1$ on A^r, where

$\mathbf{Z}/2$ acts trivially on E_0 and by -1 on E_1. It follows that

$$\begin{aligned} K_0^{\text{naive}} X &= K_0^{\text{naive}}(A^r/\mathbf{Z}/2) \\ &= \text{Rep}(\mathbf{Z}/2) \\ &= \mathbf{Z}^2. \end{aligned}$$

The true K-group $K_0 X_r$ maps isomorphically to $G_0 X_r$ since X_r is smooth over k. By the localization sequence as in Example 1, $K_0 X_r$ is the quotient of $K_0^{\text{naive}} X_r = \text{Rep}(\mathbf{Z}/2)$ by the relation that

$$W \cdot \lambda_{-1} V = (\dim W) \cdot \lambda_{-1} V$$

for all $W \in \text{Rep}(\mathbf{Z}/2)$, where V is the representation of $\mathbf{Z}/2$ on A^r. In this example, V is the trivial representation, and so $\lambda_{-1} V = 0$. Therefore the map from $K_0^{\text{naive}} X_r$ to $K_0 X_r$ is an isomorphism, as promised.

Finally, we know that X_r does not have the resolution property by Proposition 1.3, since X_r does not have affine diagonal, using that $r \geq 2$. One can define an explicit coherent sheaf K on X_r which is not a quotient of a vector bundle, by the same formula as in Example 1.

9 How to prove the resolution property in an example: the nodal cubic

Suppose that one wishes to prove the resolution property for a stack X. The proof of Theorem 1.1 gives an idea of how to proceed. In many cases, the proof indicates how to construct a coherent sheaf C on X such that X has the resolution property if and only if the single sheaf C is a quotient of a vector bundle. One statement of this type is formulated in Lemma 9.2, below. I hope that this will be a useful way to prove the resolution property in cases of interest.

In this section, we carry the procedure out in the following example.

Proposition 9.1 *Let X be the nodal cubic over a field k, that is, $\mathbf{P}^1$ with the points 0 and ∞ identified. Let $T := G_m$ act on X in the natural way. Then the quotient stack X/T has the resolution property.*

Here X is a projective variety and so the resolution property is well known for X, but the action of T is "bad" in several ways. In particular, any T-equivariant line bundle on X has degree 0 and so there is no T-equivariant embedding of X into projective space; there is not even an ample family of T-equivariant line bundles on X. Thus Theorem 2.1, due to Thomason, does not immediately apply to show that X/T has the resolution property. The phenomenon that X has an ample family of line bundles but no ample family of T-equivariant line bundles can only occur for non-normal schemes such as this one, which is why it seemed worth finding out whether X/T has the resolution property. (Since X is not normal, Theorem 1.1 as stated does not apply to X, but the methods still work.) The fact that we will prove the resolution property in this "bad" case is encouraging for Question 3 in the introduction, proposing that the resolution property is always preserved upon taking the quotient by a linear algebraic group.

Proof of Proposition 9.1. It seems convenient to begin by considering an etale double covering Y of X, the union of two copies A and B of $\mathbf{P}^1$, with 0 in A identified with ∞ in B and ∞ in A identified with 0 in B. I will write p for the point 0 in $A \subset Y$ and q for the point ∞ in $A \subset Y$. The T-action on X lifts to a T-action on Y which commutes with the $\mathbf{Z}/2$-action (switching the two copies of $\mathbf{P}^1$ in the natural way), and so we have an isomorphism of quotient stacks $X/T \cong Y/(\mathbf{Z}/2 \times T)$. The scheme Y resembles X in that it has no ample family of T-equivariant line bundles. What suggests that Y should be easier to study than X is that unlike X, Y is at least a union of T-invariant affine open subsets, $Y - p$ and $Y - q$.

To show that the stack Y/T has the resolution property, we will use the following lemma, which isolates part of the proof of Theorem 1.1.

Lemma 9.2 *Let Y be a noetherian scheme with an action of a flat affine group scheme T of finite type over $\mathbf{Z}$ or over a field. Let $S_1, \ldots, S_r$ be closed T-invariant subschemes whose complements form an affine open covering of Y. Let C be the direct sum of the ideal sheaves $I_{S_1}, \ldots, I_{S_r}$. Then the stack Y/T has the resolution property if and only if the T-equivariant coherent sheaf C on Y is a quotient of some T-equivariant vector bundle on Y.*

Proof. To say that the stack Y/T has the resolution property means that every T-equivariant coherent sheaf on Y is a quotient of some T-equivariant vector bundle on Y. So suppose that C is a quotient of some T-equivariant vector bundle E on Y. Let W be the $GL(n)$-bundle over Y corresponding to E. Then the pullback of C to W is spanned by its global sections. By the choice of C, plus affineness of the morphism $W \to Y$, it follows that the scheme W is quasi-affine (this argument is given in more detail in the proof of Theorem 1.1). Therefore the stack $W/(T \times GL(n)) \cong Y/T$ has the resolution property by Theorem 2.1. QED

We return to the union Y of two copies of $\mathbf{P}^1$ with the action of $T = G_m$. Since Y is the union of the T-invariant affine open subsets $Y - p$ and $Y - q$, Lemma 9.2 shows that Y has the resolution property if the T-equivariant coherent sheaf $I_{S_1} \oplus I_{S_2}$ on Y is a quotient of some T-equivariant vector bundle, for some T-invariant subschemes S_1 and S_2 with support equal to p and q, respectively. By the $\mathbf{Z}/2$-symmetry of Y, it suffices to show that I_S is a quotient of a T-equivariant vector bundle on Y, for some T-invariant subscheme S with support equal to p. Replacing the bundle by its dual, it is equivalent to find a T-equivariant vector bundle $\mathcal{E}$ on Y with a T-invariant section $s : O_Y \to \mathcal{E}$ which vanishes (to any order) at p and nowhere else on Y.

To define a T-equivariant vector bundle $\mathcal{E}$ on Y, we need to define T-equivariant vector bundles E on A and F on B, together with T-equivariant isomorphisms $E|_0 \cong F|_\infty$ and $E|_\infty \cong F|_0$. Take $E = O(1) \oplus O(-1)$ and $F = O(1) \oplus O(-1)$ as vector bundles on $\mathbf{P}^1$. Define the action of T on E to be trivial on $E|_\infty$, with weight 1 on $O(1)|_0 \subset E|_0$, and weight -1 on $O(-1)|_\infty \subset E|_0$. Define the action of T on F to be trivial on $F|_0$, with weight -1 on $O(1)|_\infty \subset F|_\infty$, and weight 1 on $O(-1)|_\infty \subset F|_\infty$. Clearly there are isomorphisms of T-representations $E|_0 \cong F|_\infty$ and $E|_\infty \cong F|_0$, which we can use to define a T-equivariant vector bundle $\mathcal{E} = (E, F)$ on Y. For our purpose, we need to choose the isomorphism between the trivial 2-dimensional representations $E|_\infty$ and $F|_0$ of T so as to send the line $O(1)|_\infty \subset E|_\infty$

to the line $O(1)|_0 \subset F|_0$. With this choice, the T-vector bundle $\mathcal{E}$ on Y is not a direct sum of two T-line bundles.

The vector bundle $\mathcal{E}$ has a T-invariant section over Y (contained in the subbundle $O(1) \subset E$ over A and in the subbundle $O(1) \subset F$ over B) which vanishes at p (the image of $0 \in A \cong \mathbf{P}^1$ and of $\infty \in B \cong \mathbf{P}^1$) but nowhere else in Y. Thus, as we have explained, the dual T-vector bundle $\mathcal{E}^*$ on Y maps onto the T-coherent sheaf I_S for some subscheme S with support equal to p. By Lemma 9.2, the stack Y/T has the resolution property.

From here it is easy to deduce that the quotient stack of the nodal cubic X by T also has the resolution property, as we wanted. Namely, writing σ for the generator of the $\mathbf{Z}/2$-action on Y, $\mathcal{E}^* \oplus \sigma^*(\mathcal{E}^*)$ is a $\mathbf{Z}/2 \times T$-equivariant vector bundle on Y which maps onto the $\mathbf{Z}/2 \times T$-equivariant sheaf $I_S \oplus I_{\sigma(S)}$. So, by Lemma 9.2 again, the stack $Y/(\mathbf{Z}/2 \times T) = X/T$ has the resolution property. Proposition 9.1 is proved. QED

It is interesting to add that the proof of Theorem 1.1 works completely for X/T even though the nodal cubic X is not normal, because it is a scheme, not just an algebraic space. That proof shows that since X/T has the resolution property, it is the quotient of a quasi-affine scheme by $GL(n)$ for some n. Thus this apparently bad example of a stack turns out to have some very good properties. As mentioned at the beginning of this section, this is encouraging for Question 3 in the introduction.

References

[1] *M. Artin,* Algebraic construction of Brieskorn's resolutions, *J. Algebra* **29** (1974), 330–348.

[2] *M. Atiyah, K-theory,* Benjamin (1967).

[3] *M. Atiyah,* K-theory past and present, arXiv:math.KT/0012213.

[4] *M. Borelli,* Divisorial varieties, *Pacific J. Math.* **13** (1963), 375–388.

[5] *M. Borelli,* Some results on ampleness and divisorial schemes, *Pacific J. Math.* **23** (1967), 217–227.

[6] *H. Brenner and S. Schröer,* Ample families, multihomogeneous spectra, and algebraization of formal schemes, to appear in *Pacific J. Math.,* arXiv:math.AG/0012082.

[7] *M. Demazure and P. Gabriel, Groupes algébriques,* Masson (1970).

[8] *D. Edidin, B. Hassett, A. Kresch, and A. Vistoli,* Brauer groups and quotient stacks, *Am. J. Math.* **123** (2001), 761–777.

[9] *D. Freed, M. Hopkins, and C. Teleman,* Twisted equivariant K-theory with complex coefficients, arXiv:math.AT/0206257.

[10] *W. Fulton, Introduction to toric varieties,* Princeton (1993).

[11] *P. Gabriel,* Construction de préschémas quotient, *Schémas en groupes, I*, SGA 3, 1962/64, 250–286, Lecture Notes in Mathematics **151**, Springer (1970).

[12] *A. Grothendieck,* Éléments de géométrie algébrique, II, Étude globale élémentaire de quelques classes de morphismes, *Publ. Math. IHES* **8** (1961), 222 pp., IV, Étude locale des schémas et des morphismes de schémas, Troisième partie, *Publ. Math. IHES* **28** (1966), 255 pp.

[13] *A. Grothendieck,* Le groupe de Brauer, II, Théorie cohomologique, *Dix exposés sur la cohomologie des schémas,* 67–87, North-Holland (1968).

[14] *R. Hartshorne, Algebraic geometry,* Springer (1977).

[15] *J. Hausen,* Equivariant embeddings into smooth toric varieties, to appear in *Canad. J. Math.,* arXiv:math.AG/0012082.

[16] *H. Hironaka,* Flattening theorem in complex-analytic geometry, *Am. J. Math.* **97** (1975), 503–547.

[17] *L. Illusie,* Existence de résolutions globales, *Théorie des intersections et théorème de Riemann-Roch*, SGA 6, 1966/67, 160–221, Lecture Notes in Mathematics **225**, Springer (1971).

[18] *J.-P. Jouanolou,* Une suite exacte de Mayer-Vietoris en K-théorie algébrique, *Algebraic K-theory, I* (Seattle, 1972), 293–316, Lecture Notes in Mathematics **341**, Springer (1973).

[19] *Y. Kawamata,* Francia's flip and derived categories, arXiv:math.AG/0111041.

[20] *S. Keel and S. Mori,* Quotients by groupoids, *Ann. Math.* **145** (1997), 193–213.

[21] *A. Klyachko,* Equivariant bundles over toric varieties, *Math. USSR Izv.* **35** (1990), 337–375.

[22] *D. Knutson, Algebraic spaces,* Lecture Notes in Mathematics **203**, Springer (1971).

[23] *J. Kollár,* Projectivity of complete moduli, *J. Diff. Geo.* **32** (1990), 235–268.

[24] *J. Kollár,* Cone theorems and bug-eyed covers, *J. Alg. Geo.* **1** (1992), 293–323.

[25] *J. Kollár,* Quotient spaces modulo algebraic groups, *Ann. Math.* **145** (1997), 33–79.

[26] *G. Laumon and L. Moret-Bailly, Champs algébriques,* Springer (2000).

[27] *D. Mumford,* Towards an enumerative geometry of the moduli space of curves, *Arithmetic and geometry,* v. 2, 271–328, Birkhäuser (1983).

[28] *D. Mumford, J. Fogarty, and F. Kirwan, Geometric invariant theory,* Springer (1994).

[29] *M. Nagata,* Some questions on rational actions of groups, *Algebraic geometry* (Bombay, 1968), 323–334, Oxford (1969).

[30] *D. Quillen,* Higher algebraic K-theory, I, *Algebraic K-theory, I* (Seattle, 1972), 85–147, Lecture Notes in Mathematics **341**, Springer (1973).

[31] *I. Satake,* The Gauss-Bonnet theorem for V-manifolds, *J. Math. Soc. Japan* **9** (1957), 464–492.

[32] *S. Schröer and G. Vezzosi,* Existence of vector bundles and global resolutions for singular surfaces, arXiv:math.AG/0201128.

[33] *C. Seshadri,* Quotient spaces modulo reductive algebraic groups, *Ann. Math.* **95** (1972), 511–556.

[34] *R. Thomason,* Algebraic K-theory of group scheme actions, *Algebraic topology and algebraic K-theory,* 539–563, Princeton (1987).

[35] *R. Thomason,* Equivariant resolution, linearization, and Hilbert's fourteenth problem over arbitrary base schemes, *Adv. Math.* **65** (1987), 16–34.

[36] *R. Thomason,* Equivariant algebraic vs. topological K-homology Atiyah-Segal style, *Duke Math. J.* **56** (1988), 589–636.

[37] *R. Thomason and T. Trobaugh,* Higher algebraic K-theory of schemes and of derived categories, *The Grothendieck Festschrift*, v. 3, ed. P. Cartier et al., 247–436, Birkhäuser (1990).

[38] *A. Vistoli,* Intersection theory on algebraic stacks and on their moduli spaces, *Invent. Math.* **97** (1989), 613–670.

[39] *C. Voisin,* A counterexample to the Hodge conjecture for Kähler varieties, arXiv:math.AG/0112247.

DPMMS, Wilberforce Road, Cambridge CB3 0WB, England.
b.totaro@dpmms.cam.ac.uk