Moving codimension-one subvarieties over finite fields

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In topology, the normal bundle of a submanifold determines a neighborhood of the submanifold up to isomorphism. In particular, the normal bundle of a codimension-one submanifold is trivial if and only if the submanifold can be moved in a family of disjoint submanifolds. In algebraic geometry, however, there are higher-order obstructions to moving a given subvariety.

In this paper, we develop an obstruction theory, in the spirit of homotopy theory, which gives some control over when a codimension-one subvariety moves in a family of disjoint subvarieties. Even if a subvariety does not move in a family, some positive multiple of it may. We find a pattern linking the infinitely many obstructions to moving higher and higher multiples of a given subvariety. As an application, we find the first examples of line bundles L on smooth projective varieties over finite fields which are nef (L has nonnegative degree on every curve) but not semi-ample (no positive power of L is spanned by its global sections). This answers questions by Keel and Mumford.

Determining which line bundles are spanned by their global sections, or more generally are semi-ample, is a fundamental issue in algebraic geometry. If a line bundle L is semi-ample, then the powers of L determine a morphism from the given variety onto some projective variety. One of the main problems of the minimal model program, the abundance conjecture, predicts that a variety with nef canonical bundle should have semi-ample canonical bundle [15, Conjecture 3.12].

One can hope to get more insight into the abundance conjecture by reducing varieties in characteristic zero to varieties over finite fields, where they become simpler in some ways. In particular, by Artin, every nef line bundle L with $L^2 > 0$ on a projective surface over the algebraic closure of a finite field is semi-ample [2, proof of Theorem 2.9(B)]. This is far from true over other algebraically closed fields, by Zariski [26, section 2]. Keel generalized Artin's theorem, giving powerful sufficient conditions for a nef line bundle on a projective variety over $\overline{\mathbf{F}}_p$ to be semi-ample [12, 13]. As an application, he constructed contractions of the moduli space of stable curves which exist as projective varieties in every finite characteristic, but not in characteristic zero.

Keel asked whether a nef line bundle L on a smooth projective surface over $\overline{\mathbf{F}}_p$ is always semi-ample, the open case being line bundles with $L^2 = 0$. (This is part of his Question 0.8.2 [13], in view of Theorem 3.3 below.)

Using our obstruction theory, we can see where nef line bundles that are not semi-ample should be expected and produce them. We obtain the first known examples of nef but not semi-ample line bundles on smooth projective varieties over $\overline{\mathbf{F}}_p$, for any prime number p (Theorem 6.1). Equivalently, we give faces of the closed cone of curves which have rational slope but cannot be contracted. The line bundles we construct are effective, of the form $O(C)$ for a smooth curve C of genus

2 with self-intersection zero on a smooth projective surface. Thus we give the first examples over $\overline{\mathbf{F}}_p$ of a curve with self-intersection zero such that no multiple of the curve moves in a family of disjoint curves (which would give a fibration of the surface over a curve). This answers a question raised by Mumford [20, p. 336].

One can still hope for further positive results about semi-ampleness over finite fields. Sakai [22] and Maşek [16] gave a positive result when the curve C has genus at most 1; we give Keel's proof of their result in Theorem 2.1. (This makes sense in terms of minimal model theory, since a curve C with $C^2 = 0$ in a surface X has genus at most 1 exactly when $K_X \cdot C \leq 0$.)

Building upon our basic example, we exhibit a nef *and big* line bundle on a smooth projective 3-fold over $\overline{\mathbf{F}}_p$ which is not semi-ample (Theorem 7.1).

The following question of Keel's remains open and very interesting [13, Question 0.9]. Let X be a smooth projective surface over $\overline{\mathbf{F}}_p$ with a line bundle L. If $L \cdot C > 0$ for every curve C, does it follow that L is ample (or equivalently, that $L^2 > 0$)? Counterexamples to this statement, using ruled surfaces, were given over the complex numbers by Mumford [8, Example 10.6], and over uncountable algebraically closed fields of positive characteristic by Mehta and Subramanian [17, Remark 3.2].

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1 Notation

Varieties are reduced and irreducible by definition. A *curve* on a variety means a closed subvariety of dimension 1. A line bundle L on an n-dimensional proper variety X over a field is *nef* if the intersection number $L \cdot C$ is nonnegative for every curve C in X. We often use additive notation for line bundles, in which the line bundle $L^{\otimes a}$ is called aL for an integer a. For a nef line bundle L, a curve C is called L-*exceptional* if $L \cdot C = 0$. A line bundle L is *big* if the rational map associated to some positive multiple of L is birational. We use the following fact [14, Theorem VI.2.15].

Lemma 1.1 *Let L be a nef line bundle on a proper variety of dimension n over a field. Then L is big if and only if $L^n > 0$.*

A line bundle L is *semi-ample* if some positive multiple of L is spanned by its global sections. A semi-ample line bundle is nef. Also, a semi-ample line bundle determines a contraction: the algebra $R(X, L) = \oplus_{a \geq 0} H^0(X, aL)$ is finitely generated by Zariski [26], $Y := \text{Proj } R(X, L)$ is a projective variety, and there is a natural surjection f from X to Y with connected fibers (meaning that $f_* O_X = O_Y$). Conversely, any surjective morphism with connected fibers from X to a projective variety Y arises in this way from some semi-ample line bundle L on X, by taking L to be the pullback of any ample line bundle on Y.

Mourougane and Russo gave a useful decomposition of semi-ampleness into two properties, extending earlier results by Zariski [26] and Kawamata [11]. Let L be a nef line bundle on a variety X. Define the *numerical dimension* $\nu(X, L)$ to be the largest natural number ν such that the cycle L^ν is numerically nontrivial, that is, the largest ν such that $L^\nu \cdot S > 0$ for some subvariety S of dimension ν. The *Iitaka* *dimension* $\kappa(X, L)$ is defined to be $-\infty$ if $H^0(X, aL) = 0$ for all $a > 0$. Otherwise, we define $\kappa(X, L)$ to be one less than the transcendence degree of the quotient field of the graded algebra $R(X, L)$, or equivalently the maximum dimension of the image of X under the rational maps to projective space defined by powers of L. We always have $\kappa(X, L) \leq \nu(X, L)$, and L is called *good* if the two dimensions are equal.

Theorem 1.2 *[19, Corollary 1] Let X be a normal proper variety over a field k. A nef line bundle L on X is semi-ample if and only if it is good and the graded algebra $R(X, L)$ is finitely generated over k. The finite generation is automatic when $\kappa(X, L) \leq 1$.*

For example, when $L^n > 0$, in other words when L has maximal numerical dimension, Lemma 1.1 says that L is automatically good, and so semi-ampleness is purely a question of finite generation. This paper is about the "opposite" situation, where L has numerical dimension 1; by Theorem 1.2, semi-ampleness is equivalent to goodness in this case. On the other hand, problems of semi-ampleness always have some relation to problems of finite generation. For example, if L is a nef line bundle on a projective variety X and M is an ample line bundle, then L is semi-ample if and only if the algebra $\oplus_{a,b \geq 0} H^0(X, aL + bM)$ is finitely generated, as one easily checks.

Finally, here is one last standard result.

Lemma 1.3 *Every numerically trivial line bundle on a proper scheme over the algebraic closure of a finite field is torsion.*

The proof is based on the fact that an abelian variety has only finitely many rational points over any given finite field. A reference is [12, Lemma 2.16].

2 Moving elliptic curves on surfaces over finite fields

Sakai [22, Theorem 1, Proposition 5, and Concluding Remark] and Maşek [16, Theorem 1 and Lemma] showed that for any curve of genus 1 with self-intersection zero on a smooth projective surface over $\overline{\mathbf{F}}_p$, some positive multiple of the curve always moves. In this section we give Keel's proof of this result. The corresponding statement is easy for curves of genus 0 on a surface (over any field, in fact) and false for genus at least 2 (Theorem 6.1).

Theorem 2.1 *Let C be a curve of arithmetic genus 1 in a smooth projective surface X over $\overline{\mathbf{F}}_p$. If $C^2 = 0$, then $L = O(C)$ is semi-ample. Equivalently, C is a fiber (possibly a multiple fiber) in some elliptic or quasi-elliptic fibration of X.*

This is false over all algebraically closed fields k except the algebraic closure of a finite field, by some standard examples. First, let C be a smooth cubic curve in $\mathbf{P}^2$, and let X be the blow-up of $\mathbf{P}^2$ at 9 points on the curve. Since $C^2 = 9$ in $\mathbf{P}^2$, the proper transform C in X has $C^2 = 0$. By appropriate choice of the points to blow up, we can make the normal bundle of C in X any line bundle of degree 0 on C. If k is not the algebraic closure of a finite field, this normal bundle can be non-torsion in the Picard group of C, and so no positive multiple of C moves in

X. Another example, on a ruled surface, works only in characteristic zero: see the discussion after Lemma 4.1.

Proof. Since the normal bundle $N_{C/X} = L|_C$ has degree 0, it is torsion by Lemma 1.3 since the base field k is $\overline{\mathbf{F}}_p$. Let m be the order of $L|_C$ in $\text{Pic}(C)$. For any integer a, we have the exact sequence:

$$H^0(C, aL) \to H^1(X, (a-1)L) \to H^1(X, aL) \to H^1(C, aL).$$

Since C has genus 1 and the line bundle aL is nontrivial on C for $1 \leq a \leq m-1$, we have $H^0(C, aL) = 0$ and $H^1(C, aL) = 0$ in that range. So the exact sequence gives isomorphisms

$$H^1(X, O) \cong H^1(X, L) \cong \cdots \cong H^1(X, (m-1)L).$$

Next, we have the exact sequence

$$0 \to H^0(X, (m-1)L) \to H^0(X, mL) \to H^0(C, mL) \to H^1(X, (m-1)L).$$

Here $H^0(C, mL) \cong k$. Thus, if $H^1(X, O)$ is zero, then $H^1(X, (m-1)L)$ is zero, and so the sequence shows that the divisor mC moves nontrivially in X. Therefore $\kappa(X, L) \geq 1$, and L is semi-ample by Theorem 1.2.

In general, we use the idea of "killing cohomology": for any projective variety X over a field k of characteristic $p > 0$, and any element α of $H^1(X, O)$, there is a surjective morphism $f : W \to X$ of projective varieties which kills α, meaning that $f^*(\alpha) = 0$. Indeed, by Serre, for X smooth one can do this with a finite flat morphism f (a composite of etale $\mathbf{Z}/p$-coverings and Frobenius morphisms) [23, Proposition 12 and section 9]. The construction shows that W is smooth since X is. Then the above proof applied to W shows that $\kappa(W, f^*(L)) \geq 1$. By Ueno, for any surjective morphism f of normal projective varieties, we have $\kappa(W, f^*(L)) = \kappa(X, L)$ [25, Theorem 5.13]. Thus $\kappa(X, L) \geq 1$ and so L is semi-ample. QED

3 L-equivalence

In this section, we show that L-equivalence (as defined by Keel) is automatically bounded on a normal projective surface over any field. The definition is given before Theorem 3.3. (By contrast, Kollár observed that L-equivalence can be unbounded on a non-normal surface [13, section 5].) The proof is elementary geometry of surfaces (the Hodge index theorem).

The "reduction map for nef line bundles" of Bauer et al. [3] is a similar application of the Hodge index theorem which works in any dimension, but some work would be needed to go from their theorem to the boundedness of L-equivalence on normal surfaces. Their theorem is stated over the complex numbers, but the argument works in any characteristic. The difficulty is that their theorem only applies to "general" points, meaning points outside a countable union of proper subvarieties [3, Theorem 2.1, 2.4.2]. For example, it seems to be unknown whether there is a nef line bundle L on some normal complex projective variety X such that the set of curves C with $L \cdot C = 0$ is countably infinite. That does not happen for X of dimension 2 by Lemma 3.1 below and [24, Theorem 2.1], but the latter result uses the Albanese map in a way that has no obvious generalization to higher dimensions.

Let $\rho(X)$ denote the Picard number of a projective variety X, that is, the dimension of the real vector space $N^1(X)$ generated by line bundles modulo numerical equivalence.

Lemma 3.1 *Let X be a smooth projective surface over a field. Let L be a nef line bundle on X such that $L^2 = 0$ and L is numerically nontrivial. Then there are at most $2(\rho(X) - 2)$ curves A on X such that $L \cdot A = 0$ and $A \notin \mathbf{R}^{>0} \cdot L \subset N^1(X)$. (These curves A will all have negative self-intersection.)*

If in addition there is no effective 1-cycle in $\mathbf{R}^{>0} \cdot L$, or if there is a curve C in X such that every effective 1-cycle in $\mathbf{R}^{>0} \cdot L$ is a multiple of C as a cycle, then there are at most $\rho(X) - 2$ curves $A \neq C$ with $L \cdot A = 0$.

Finally, let L be a nef line bundle on a surface with $L^2 > 0$. Then there are at most $\rho(X) - 1$ curves A with $L \cdot A = 0$.

All these bounds are optimal, as we now check.

Example. Let X be the blow-up of $\mathbf{P}^1 \times \mathbf{P}^1$ at d points whose projections to the second factor are distinct. Then X has Picard number $d + 2$. Let L be the pullback to X of the line bundle $O(1)$ on the second factor of $\mathbf{P}^1$. Then there are exactly $2d$ curves A with $L \cdot A = 0$ which are not in the ray $\mathbf{R}^{>0} \cdot L \subset N^1(X)$, namely the proper transforms of the fibers containing the d given points, together with the d exceptional curves. These $2d$ curves are all (-1)-curves. This shows the optimality of the first statement in Lemma 3.1.

Next, assume that the base field has characteristic zero, and let X be the ruled surface over a curve of genus at least 1 associated to a nontrivial extension of the trivial line bundle by itself. Let C be the section in X with zero self-intersection. Then every curve in the ray $\mathbf{R}^{>0} \cdot C \subset N^1(X)$ is a multiple of C as a cycle. In this case, X has Picard number 2 and, by Lemma 3.1, there is no curve $A \neq C$ with $C \cdot A = 0$. Blowing up X at d points not on C gives a surface M with Picard number $d+2$ such that the line bundle $L = O(C)$ on M has exactly d curves $A \neq C$ with $C \cdot A = 0$, namely the d exceptional curves. This shows the optimality of the second statement in Lemma 3.1.

Finally, let X be the blow-up of $\mathbf{P}^2$ at d points, and let L be the pullback of the line bundle $O(1)$ to X, which is nef and big. Then X has Picard number $d+1$, and there are exactly d curves on which L has degree zero, namely the d exceptional curves. This shows the optimality of the last statement of the lemma.

Proof of Lemma 3.1. We have that $L^2 = 0$ and L is numerically nontrivial. By the Hodge index theorem [7, Theorem V.1.9], the intersection form on $N^1(X)$ passes to a negative definite form on $V := L^{\perp}/(\mathbf{R} \cdot L)$. As a result, every curve A in X with $L \cdot A = 0$ and $A \notin \mathbf{R}^{>0} \cdot L$ has $A^2 < 0$. Also, for any two distinct curves A_1 and A_2 with these properties, we have $A_1 \cdot A_2 \geq 0$.

We use the following elementary lemma on quadratic forms [4, section V.3.5].

Lemma 3.2 *Let V be a real vector space of dimension n with a negative definite symmetric bilinear form. Let S be a subset spanning V such that any two distinct elements of S have inner product at least 0. Suppose that S cannot be partitioned into two nonempty subsets A and B with A orthogonal to B. Then S has either n or $n + 1$ elements, and in the latter case 0 is a linear combination of the elements of S with all coefficients positive.*

By Lemma 3.2, the set S of curves A in X with $L \cdot A = 0$ and $A \notin \mathbf{R}^{>0} \cdot L$ has at most $2\dim(V)$ elements (the extreme case being when $S \subset V$ is a union of $\dim(V)$ pairs of vectors in lines orthogonal to each other). Since V has dimension $\rho(X) - 2$, S has order at most $2(\rho(X) - 2)$, as we want.

We now prove the second statement of Lemma 3.1. In this case, we have the additional information that $0 \in V := L^{\perp}/(\mathbf{R} \cdot L)$ is not in the convex hull of the set S. By Lemma 3.2, S has at most $\dim(V) = \rho(X) - 2$ elements, as we want.

For the third statement of Lemma 3.1, we have $L^2 > 0$. In this case, the Hodge index theorem gives that the intersection form on $L^{\perp}$ is negative definite. Let V denote $L^{\perp}$ in this case. Here $0 \in V$ is not in the convex hull of the set S of classes of curves A with $L \cdot A = 0$, using that X is projective. By Lemma 3.2, S has at most $\dim(V) = \rho(X) - 1$ elements. QED

We now recall Keel's notion of L-equivalence. Let L be a nef line bundle on a proper scheme X over a field. Two closed points in X are called *L-equivalent* if they can be connected by a chain of curves C such that $L \cdot C = 0$. Say that L-equivalence is *bounded* if there is a positive integer m such that any two L-equivalent points can be connected by a chain with length at most m of such curves.

Keel's Question 0.8.2 [13] asks: Given a nef line bundle L on a projective scheme X over $\overline{\mathbf{F}}_p$ such that L-equivalence is bounded, is L semi-ample? We now check that on a smooth projective surface over any field, L-equivalence is always bounded (Theorem 3.3). As a result, the examples in this paper of nef but not semi-ample line bundles on smooth projective surfaces over $\overline{\mathbf{F}}_p$ give a negative answer to Question 0.8.2.

Theorem 3.3 *For any nef line bundle L on a normal proper algebraic space X of dimension 2 over a field, L-equivalence is bounded.*

Proof. First suppose X is smooth and projective. The theorem is clear if L is numerically trivial, since any two points lie on a curve. Also, if L is big ($L^2 > 0$), then there are only finitely many L-exceptional curves by Lemma 3.1.

So we can assume that $L^2 = 0$ and L is numerically nontrivial. By Lemma 3.1, there are only finitely many L-exceptional curves A which are not numerically equivalent to a multiple of L. There may be infinitely many curves C which are numerically equivalent to a multiple of L, but every such curve is clearly disjoint from all other L-exceptional curves. Thus L-equivalence is bounded.

Now let X be normal, not necessarily smooth. There is a resolution of singularities $f : M \to X$ with M projective. The $f^*(L)$-exceptional curves on M are the proper transforms of the L-exceptional curves together with the finitely many curves contracted by f. This makes it clear that L-equivalence is bounded on X in the cases where L is numerically trivial or big. For $L^2 = 0$ and L numerically nontrivial, the previous paragraph's results applied to $f^*(L)$ on M imply that all but finitely many L-exceptional curves of X are disjoint from the singular points of X and from all other L-exceptional curves; so again L-equivalence is bounded. QED

4 General remarks on moving divisors

Here we discuss the obstructions to moving codimension-one subvarieties in general terms, to motivate our main technical result, Theorem 5.1. The main application in this paper is to construct the first examples of nef line bundles on smooth projective varieties over finite fields which are not semi-ample.

We want to find a curve C with self-intersection zero in some surface M over $\overline{\mathbf{F}}_p$ such that no multiple of C moves on M. In order for this to happen, C must have arithmetic genus at least 2 by Theorem 2.1. Moreover, the normal bundle of C must be nontrivial by Lemma 4.1, stated by Mumford and discussed below. On the other hand, the normal bundle of C must be torsion in the Picard group of C, because it is a line bundle of degree zero over $\overline{\mathbf{F}}_p$ (Lemma 1.3). We might therefore look for counterexamples in what seems to be the simplest remaining case: where C is a smooth curve of genus 2 whose normal bundle has order 2 in the Picard group of C. There are indeed counterexamples of this type over $\overline{\mathbf{F}}_p$ for any prime number p, but it turns out to be simpler to give counterexamples where C is a curve of genus 2 with normal bundle of order p equal to the characteristic. We give only the latter type of counterexample in this paper.

The examples we give are curves of genus 2 in rational surfaces M, obtained by blowing up $\mathbf{P}^1 \times \mathbf{P}^1$ at 12 points on a smooth curve of genus 2 (so that the proper transform C has $C^2 = 0$). It is easy to choose the points we blow up to ensure that the normal bundle of C has order p in the Picard group of C. Theorem 5.1 gives a Zariski open condition (on the family of such arrangements of points and curves) which implies that no multiple of C moves on M. That is the key point; a priori, to prove that no multiple of C moves would require us to check infinitely many open conditions. In Theorem 6.1, we show that there are indeed surfaces of this type that have nef line bundles which are not semi-ample, over $\overline{\mathbf{F}}_p$ for every prime number p.

Once we have a rational surface over $\overline{\mathbf{F}}_p$ which has a nef line bundle which is not semi-ample, we get such line bundles on many other surfaces over $\overline{\mathbf{F}}_p$, including surfaces of general type. Take any smooth projective surface S with a surjective morphism $f : S \to M$; for example, take a ramified covering of M and resolve singularities. Let L be a nef line bundle on M which is numerically nontrivial but has Iitaka dimension $\kappa(M, L) \leq 0$, as in our examples. Then the pullback line bundle f^*L has the same properties. Indeed, we have $\kappa(S, f^*L) = \kappa(M, L)$ for every surjective morphism $f : S \to M$ of normal varieties, by Ueno [25, Theorem 5.13]. Thus the line bundle f^*L is nef but not semi-ample on the surface S.

Here is the lemma stated by Mumford, as mentioned above. He left the proof as a "curiosity for the reader" [20, Proposition, p. 336]. A proof was given by Maşek [16, Lemma, p. 682]. The lemma explains our use of curves with nontrivial but torsion normal bundle in order to construct nef line bundles over finite fields which are not semi-ample, but it will not actually be used in the rest of the paper. We will state and prove the lemma in any dimension, using the idea of killing cohomology as in the proof of Theorem 2.1.

Lemma 4.1 *Let D be an effective Cartier divisor in a normal projective variety X over a field of characteristic $p > 0$, and let L be the line bundle $O(D)$ on X. Assume that the restriction of L to D, the normal bundle of D in X, has finite order m in the Picard group of D (so in particular the normal bundle is numerically trivial).*

Then L is semi-ample on X (so that some positive multiple of D moves in its linear system with no base points) if and only if the line bundle mp^rL is trivial on the subscheme mp^rD of X for some $r \geq 0$.

Here, for any effective Cartier divisor D in a normal scheme X, D will be defined locally by one equation $f = 0$; then aD denotes the subscheme of X defined locally by $f^a = 0$, for any positive integer a. For any integer a, we will write aL for the line bundle $L^{\otimes a} = O(aD)$ on M.

In characteristic zero, it is not true that the triviality of the line bundle mL on the subscheme mD implies that some multiple of D moves in its linear system. A counterexample is provided by the $\mathbf{P}^1$-bundle over an elliptic curve associated to a nontrivial extension of the trivial line bundle by itself. Here a section D is an elliptic curve with trivial normal bundle (that is, $L := O(D)$ is trivial on D) and yet no multiple of D moves. But there is an analogous result in characteristic zero, by Neeman. Let D be a curve in a smooth projective surface X over a field k of characteristic zero such that the normal bundle has finite order m. Then some positive multiple of D moves in its linear system on X (or, equivalently, $L = O(D)$ is semi-ample on X) if and only if the line bundle mL is trivial on the subscheme $(m + 1)D$ [21, Article 2, Theorem 5.1]. Neeman works over the complex numbers and considers smooth curves, but only minor changes to the argument are needed to avoid those restrictions.

Proof of Lemma 4.1. First suppose that L is semi-ample on X. Then aL is basepoint-free on X for some positive integer a, and hence aL is basepoint-free on the subscheme aD. But L is also numerically trivial on aD, and so aL is trivial on aD. Clearly a must be a multiple of m, and so we can write $a = mp^rj$ for some $r \geq 0$ and some positive integer j which is not a multiple of p. The group ker $(\text{Pic}(mp^rjD) \to \text{Pic}(D))$ is p-primary by the exact sequence used in the proof of Lemma 5.2 below. Since the line bundle mp^rL is trivial on D and mp^rjL is trivial on mp^rjD, it follows that mp^rL is trivial on mp^rjD. A fortiori, mp^rL is trivial on mp^rD.

Conversely, suppose that aL is trivial on aD for $a = mp^r$ and some $r \geq 0$. By Lemma 1.2, to show that L is semi-ample on X, it suffices to show that $\kappa(X, L)$ is at least 1, which holds if $H^0(X, nL)$ has dimension at least 2 for some positive integer n. By the exact sequence

$$0 \to O_X \to O_X(aD) \to O_X(aD)|_{aD} \to 0$$

of sheaves on X, we have an exact sequence of cohomology groups

$$H^0(X, O) \to H^0(X, aL) \to H^0(aD, aL) \to H^1(X, O).$$

Since aL is trivial on aD, we have $h^0(X, aL) \geq 2$ and hence L is semi-ample if $H^1(X, O) = 0$. In general, we use the idea of killing cohomology as in the proof of Theorem 2.1. We find that there is a normal variety Y with a finite morphism $f : Y \to X$ such that the obstruction class in $H^1(X, O)$ pulls back to 0 in $H^1(Y, O)$. Therefore $\kappa(Y, f^*L) \geq 1$. By Ueno [25, Theorem 5.13], it follows that $\kappa(X, L) \geq 1$. Thus L is semi-ample. QED

5 Obstruction theory for moving divisors

In this section, we prove our main general result of obstruction theory for moving divisors, Theorem 5.1. It gives a sufficient condition on a divisor D with torsion normal bundle in a variety X so that no multiple of D moves. (If the normal bundle is numerically trivial but not torsion, then it is clear that no multiple of D moves.) The condition is Zariski open on the family of divisors with normal bundle of a given order. Section 6 will apply this theorem to give nef line bundles which are not semi-ample on smooth projective varieties over finite fields.

Theorem 5.1 *Let D be a connected reduced Cartier divisor in a normal projective variety X over an algebraically closed field of characteristic $p > 0$. Let L denote the line bundle $O(D)$ on X. Suppose that the restriction of L to D has order equal to p in the Picard group of D (so in particular D has numerically trivial normal bundle). Suppose that the line bundle pL is nontrivial on the subscheme $2D$ of X. Finally, suppose that the Frobenius maps $F^* : H^1(D, -L) \to H^1(D, -pL) \cong H^1(D, O)$ and $F^* : H^1(D, O) \to H^1(D, O)$ are injective.*

Then L is not semi-ample on X. More precisely, no multiple of D moves in its linear system on X.

Proof. In view of the inclusions $H^0(X, O(D)) \subset H^0(X, O(2D)) \subset \cdots$, it suffices to show that $H^0(X, O(p^{r+1}D))$ has dimension 1 for all $r \geq 0$. By the exact sequence of sheaves

$$0 \to O_X \to O_X(p^{r+1}D) \to O_X(p^{r+1}D)|_{p^{r+1}D} \to 0,$$

it suffices to show that $H^0(p^{r+1}D, p^{r+1}L) = 0$ for all $r \geq 0$. That will be accomplished by Lemma 5.2. The proof of Lemma 5.2 will require a simultaneous induction to show that every regular function on $p^{r+1}D$ is constant and that $p^{r+1}L$ is nontrivial on $p^{r+1}D$. QED

Lemma 5.2 *We retain the assumptions of Theorem 5.1. For any integer j and any $r \geq 0$, the group $H^0(p^rD, -p^rjL)$ is 0 for j not a multiple of p, and it injects into $H^0(D, -p^rjL) \cong k$ for j a multiple of p. Also, the line bundle $p^{r+1}L$ is nontrivial on $(p^r + 1)D$ (and hence on $p^{r+1}D$).*

Proof. Since D is connected and reduced, the group $H^0(D, O)$ of regular functions on D is equal to the algebraically closed base field k. For $r = 0$, the first statement of the lemma holds because L is a line bundle of order equal to p in the Picard group of the divisor D. Also, pL is nontrivial on $(p^0 + 1)D = 2D$ by assumption.

For any positive integer a, we have an exact sequence of sheaves supported on D,

$$0 \to L^{\otimes -a}|_D \to O^*_{(a+1)D} \to O^*_{aD} \to 0.$$

Part of the long exact sequence on cohomology looks like:

$$H^1(D, -aL) \to \text{Pic}((a + 1)D) \to \text{Pic}(aD).$$

Since the line bundle pL is trivial on D, this sequence shows that the isomorphism class of pL on $2D$ is the image of some element η in $H^1(D, -L)$. Since pL is nontrivial on $2D$, η is not zero.

Now let $r \geq 1$ and assume the lemma for $r-1$. For any positive integers a and b, we have an exact sequence of sheaves supported on D,

$$0 \to L^{\otimes -a}|_{bD} \to O_{(a+b)D} \to O_{aD} \to 0.$$

Tensoring with a line bundle and taking cohomology gives the following exact sequence, for any $1 \leq i \leq p-1$ and any integer j:

$$0 \to H^0(p^{r-1}D, -p^{r-1}(pj+i)L) \to H^0(p^{r-1}(i+1)D, -p^r jL) \to H^0(p^{r-1}iD, -p^r jL).$$

Since $pj+i$ is not a multiple of p, the first H^0 group is zero by induction. Combining these exact sequences for $i = 1, \ldots, p-1$, we find that $H^0(p^r D, -p^r jL)$ injects into $H^0(p^{r-1}D, -p^r jL)$. By our inductive assumption again, the latter group in turn restricts injectively to $H^0(D, -p^r jL) \cong k$. This is the conclusion we want when j is a multiple of p.

For j not a multiple of p, we use that $p^r L$ is nontrivial on $(p^{r-1}+1)D$, hence on $p^r D$. The exact sequence

$$H^1(D, -aL) \to \text{Pic}((a+1)D) \to \text{Pic}(aD)$$

for positive integers a, where $H^1(D, -aL)$ is a k-vector space, shows that ker $(\text{Pic}(p^r D) \to \text{Pic}(D))$ is a p-primary group. Therefore $p^r jL$ is nontrivial on $p^r D$ for all j not a multiple of p. Since $H^0(p^r D, -p^r jL)$ injects into $H^0(D, -p^r jL) \cong H^0(D, O) \cong k$, it follows that $H^0(p^r D, -p^r jL) = 0$ (since a nonzero section would give a trivialization of $-p^r jL$ over $p^r D$). This is what we want for j not a multiple of p.

Recall that η is the nonzero element of $H^1(D, -L)$ that describes the isomorphism class of the line bundle pL on $2D$. By our assumptions on injectivity of Frobenius maps, $(F^*)^r(\eta)$ is nonzero in $H^1(D, -p^r L)$. Consider the exact sequence

$$H^0((p^r+1)D, O^*) \to H^0(p^r D, O^*) \to H^1(D, -p^r L) \to \text{Pic}((p^r+1)D).$$

We have shown (by the case $j = 0$, above) that the group $H^0(p^r D, O)$ of regular functions on $p^r D$ consists only of the constants k, and so the first arrow in this sequence is surjective. Therefore the last arrow is injective, and so the image of $(F^*)^r(\eta)$ in $\text{Pic}((p^r+1)D)$ is nonzero. This means that $p^{r+1}L$ is nontrivial on $(p^r+1)D$, completing the induction. QED

6 Curves with normal bundle of order p

Theorem 6.1 *For every prime number p, there is a smooth projective surface M over $\overline{\mathbf{F}}_p$ and a smooth curve C of genus 2 in M such that C has self-intersection zero and no positive multiple of C moves in its linear system on M. Therefore the line bundle $L = O(C)$ is nef but not semi-ample on M.*

Thus, there are nef line bundles on smooth projective varieties over finite fields which are not semi-ample, in every characteristic $p > 0$. The example here can be considered optimal, in view of the positive results listed in section 4.

The surface M will be rational. We will start with a smooth curve of genus 2 in $X = \mathbf{P}^1 \times \mathbf{P}^1$ and then blow up to make its self-intersection zero. (It seems easier to start with $\mathbf{P}^1 \times \mathbf{P}^1$ rather than $\mathbf{P}^2$, since a curve of geometric genus 2 in

$\mathbf{P}^2$ is always singular.) The following lemma is a first step. We say that a possibly singular curve C over a field of positive characteristic is *ordinary* if the Frobenius map $F^* : H^1(C, O) \to H^1(C, O)$ is injective.

Lemma 6.2 *Let C be a smooth curve of bidegree $(2, 3)$ in $\mathbf{P}^1 \times \mathbf{P}^1$ (so C has genus 2) over an algebraically closed field of characteristic $p > 0$. Suppose that C is ordinary. Then there is an effective divisor D of degree 12 on C with the following property. Let M be the blow-up of $\mathbf{P}^1 \times \mathbf{P}^1$ at the 12 points of D, and let L be the line bundle on M associated to the proper transform of C. Then L (restricted to C) has order p on C, while the line bundle $O(pC) = pL$ is nontrivial on the subscheme $2C$ of M.*

Proof. We first summarize the argument. We start with a curve C in $\mathbf{P}^1 \times \mathbf{P}^1$ with genus 2 and self-intersection number 12. There are many ways to choose 12 points on C to blow up in order to make the normal bundle of C any degree-zero line bundle we want, in particular to make it a line bundle of order p. The hard part is to arrange that p times the line bundle $L := O(C)$ on the blown-up surface M is nontrivial on the subscheme $2C$. We use a formula by Illusie (Lemma 6.3) to describe the isomorphism class of pL on $2C$ as a cup product. So we have to show that certain cup products of cohomology classes on C are nonzero, which ultimately reduces to Castelnuovo's theorem on surjectivity of multiplication maps on curves.

As a first step, we have to analyze the tangent bundle of $X = \mathbf{P}^1 \times \mathbf{P}^1$ restricted to C. This restriction is an extension of two line bundles, $0 \to TC \to TX|_C \to N_{C/X} \to 0$. Let $\beta \in H^1(C, TC - N_{C/X})$ be the class of this extension. I claim that β is highly nontrivial in the sense that for each effective divisor B of degree 4 on C, the image of β in $H^1(C, TC - N_{C/X} + B)$ is nonzero.

We can view this image as the class of the extension $0 \to TC \to E \to N_{C/X}(-B) \to 0$, where E is a subsheaf of $TX|_C$ in an obvious way. If this extension splits, then we have a nonzero map from the line bundle $N_{C/X}(-B)$ to E and hence to $TX|_C$. Here $N_{C/X}$ has degree 12 and so $N_{C/X} - B$ is a line bundle of degree 8. On the other hand, since C has bidegree $(2, 3)$ and $T\mathbf{P}^1$ has degree 2, the restriction of $TX = T\mathbf{P}^1 \oplus T\mathbf{P}^1$ to C is the direct sum of two line bundles of degrees 4 and 6. So any map from a line bundle of degree 8 to $TX|_C$ is zero. Thus the extension class β in $H^1(C, TC - N_{C/X})$ is highly nontrivial in the sense claimed.

For the moment, fix a line bundle L of order p on C. (We know that the p-torsion subgroup of $\mathrm{Pic}(C)$ is isomorphic to $(\mathbf{Z}/p)^2$, because C is ordinary of genus 2.) We want to blow up $X = \mathbf{P}^1 \times \mathbf{P}^1$ at a divisor D of degree 12 on C so that the normal bundle of C in the blow-up M is isomorphic to L. Here $L = N_{C/M} = N_{C/X} - D$, and so D must be the divisor of a nonzero section $\delta \in H^0(C, N_{C/X} - L)$. The line bundle $N_{C/X} - L$ has degree 12, and so this H^0 group has dimension 11 by Riemann-Roch; thus there are many possible divisors D such that $N_{C/M}$ is isomorphic to L.

The restriction of the tangent bundle of the blow-up M to C is the extension

$$0 \to TC \to TM|_C \to N_{C/X}(-D) \to 0$$

which is the restriction of the extension $0 \to TC \to TX|_C \to N_{C/X} \to 0$ denoted β. (That is, we can view $TM|_C$ as a subsheaf of $TX|_C$ using the derivative of the map $M \to X$.) Equivalently, in terms of an identification $L \cong N_{C/X}(-D)$, we can describe the class of the extension $TM|_C$ in $H^1(C, TC - L)$ as the image under the cup product

$$H^1(C, TC - N_{C/X}) \otimes H^0(C, N_{C/X} - L) \to H^1(C, TC - L)$$

of (β, δ).

Knowing the class of $TM|_C$ as an extension (the Kodaira-Spencer class) amounts to knowing the isomorphism class of the subscheme $2C$ of M. Some references are Morrow-Rossi [18, Theorem 2.5] or, in greater generality, Illusie [9, Theorem 1.5.1]. Therefore, we can hope to describe the class of the line bundle pL on $2C$ in terms of this extension, and this is accomplished by Lemma 6.3, a consequence of Illusie's results on deformation theory. (Here we write L for the line bundle $O(C)$ on M. In fact, the line bundle pL on $2C$ only depends on the line bundle L on C, because pL is the Frobenius pullback $F^*(L)$ via the map $F : 2C \to C$.) Since L has order p on C, the class of pL on $2C$ lies in ker $(\text{Pic}(2C) \to \text{Pic}(C)) \cong H^1(C, -L)$, by the exact sequence in the proof of Lemma 5.2. We will use Cartier's theorem that, for any smooth proper variety X over an algebraically closed field k of characteristic $p > 0$, the p-torsion subgroup $\text{Pic}(X)[p]$ tensored with k injects naturally into $H^0(X, \Omega^1)$ [10, 6.14.3]. Thus, for a curve C, $\text{Pic}(C)[p]$ tensored with k injects into $H^0(C, K_C)$.

We first recall Illusie's definition of the map $\text{Pic}(X)[p] \to H^0(X, \Omega^1)$ [10, 6.14]. We are given a line bundle L with a trivialization of $L^{\otimes p}$. Let $\{U_i\}$ be an open covering of Y on which we choose trivializations of L. Let $e_{ij} \in O(U_i \cap U_j)^*$ be the transition functions; then the trivialization of $L^{\otimes p}$ gives functions $u_i \in O(U_i)^*$ such that $e_{ij}^p = u_j/u_i$. It follows that $d \log u_j - d \log u_i = p \, d \log e_{ij} = 0$, and so the 1-forms $d \log u_i$ fit together to give the desired class in $H^0(Y, \Omega^1)$.

The following lemma separates the two roles of L in our problem: a line bundle of order p on C and the normal bundle of C in M. This yields a clearer and more general statement.

Lemma 6.3 *Let C be a smooth compact subvariety of a smooth variety M over an algebraically closed field of characteristic $p > 0$. Let L be a line bundle on M (or just on $2C$) such that pL is trivial on C. Then the class of the line bundle pL in ker $(Pic(2C) \to Pic(C)) \cong H^1(C, N^*_{C/M})$ is the cup product of the class of L in $H^0(C, \Omega^1)$ with the class of the extension $TM|_C$ in $H^1(C, TC \otimes N^*_{C/M})$:*

$$H^0(C, \Omega^1_C) \otimes H^1(C, TC \otimes N^*_{C/M}) \to H^1(C, N^*_{C/M}).$$

Proof. This can be proved by an explicit cocycle calculation, but we will instead deduce it from Illusie's general results. Namely, let G be a flat group scheme over a scheme S. For any extension $Y \hookrightarrow Y'$ of a scheme Y over S by a square-zero ideal sheaf, the obstruction to extending a G-torsor over Y (in the fpqc topology) to Y' is the product of the Atiyah class of the G-torsor over Y with the Kodaira-Spencer class of the extension $Y \hookrightarrow Y'$ [9, 2.7.2].

Apply this to the group scheme $G = \mu_p$ of pth roots of unity over a field k of characteristic $p > 0$. Using the exact sequence

$$H^1(Y, \mu_p) \longrightarrow H^1(Y, G_m) \xrightarrow[p]{} H^1(Y, G_m),$$

we can rephrase the lemma in terms of the obstruction to extending a μ_p-torsor from $Y = C$ to $Y' = 2C$. The analogue of the dual of the Lie algebra of μ_p in Illusie's theory is the object $\mathfrak{g}^* = k[1] \oplus k$ of the derived category of k [9, Example 4.3.4]. This calculation uses that μ_p is a codimension-one subgroup of the one-dimensional group scheme G_m. As a result, the Atiyah class of a μ_p-torsor on a smooth scheme Y over k lies in $\text{Ext}^1_Y(\mathfrak{g}^*_Y, \Omega^1_Y) = H^0(Y, \Omega^1) \oplus H^1(Y, \Omega^1)$. The Kodaira-Spencer class of the extension $Y \hookrightarrow Y'$ lies in $H^1(Y, TY \otimes N^*_{Y/Y'})$. Finally, the obstruction to extending a μ_p-torsor lies in $H^1(Y, N^*_{Y/Y'}) \oplus H^2(Y, N^*_{Y/Y'})$. The second part of this obstruction is the obstruction to extending the line bundle L from C to $2C$, which is zero since we are given a line bundle L on $2C$. The first part of this obstruction is computed by Illusie's product formula. QED (Lemma 6.3).

We now return to the blow-up M of $X = \mathbf{P}^1 \times \mathbf{P}^1$. Let β in $H^1(C, TC - N_{C/X})$ be the class of the extension $TX|_C$, and let δ in $H^0(C, N_{C/X} - L)$ be a section whose zero set is the divisor on C where we blow up. Let γ be the class of the p-torsion line bundle L in $H^0(C, K_C)$. By Lemma 6.3 together with our earlier results, the class of pL in ker $(\text{Pic}(2C) \to \text{Pic}(C)) = H^1(C, -L)$ is the product $\beta\delta\gamma$ in:

$$H^1(C, TC - N_{C/X}) \otimes H^0(C, N_{C/X} - L) \otimes H^0(C, K_C) \to H^1(C, -L).$$

The lemma is proved if there is a line bundle L on C of order p (which determines γ) and a section $\delta \in H^0(C, N_{C/X} - L)$ such that the product $\beta\delta\gamma$ in $H^1(C, -L)$ is not zero. By Serre duality, this product becomes:

$$H^0(C, N_{C/X} - L) \otimes H^0(C, K_C) \otimes H^0(C, K_C + L) \to H^0(C, 2K_C + N_{C/X}) \xrightarrow{\beta} k.$$

We want to show: (*) for some line bundle L on C of order p (which determines γ), some $\delta \in H^0(C, N_{C/X} - L)$, and some $\alpha \in H^0(C, K_C + L)$, the product $\beta(\delta\gamma\alpha) \in k$ is not zero.

Here L is a nontrivial line bundle of degree 0 on the curve C of genus 2, and so $h^0(C, K_C + L) = 1$ by Serre duality. Let B_L be the base locus of $K_C + L$ on C, that is, the zero locus of a nonzero section of $K_C + L$; clearly B_L is an effective divisor of degree 2 on C, since $K_C + L$ has degree 2. Then it is clear that the image of the product

$$H^0(C, N_{C/X} - L) \otimes H^0(C, K_C + L) \to H^0(C, K_C + N_{C/X})$$

is the subspace $H^0(C, K_C + N_{C/X} - B_L) \subset H^0(C, K_C + N_{C/X})$ of sections that vanish on B_L.

We can now use the assumption that C is ordinary to deduce that the finite group $\text{Pic}(C)[p] \cong (\mathbf{Z}/p)^2$ spans the k-vector space $H^0(C, K_C) \cong k^2$. Let L_1 and L_2 be two line bundles of order p on C whose classes span $H^0(C, K_C)$. Let B_1 and B_2 be the base loci of $K_X + L_1$ and $K_X + L_2$, respectively, which are effective divisors of degree 2 on C. Then $B := B_1 + B_2$ is an effective divisor of degree 4 on C. For $i = 1$ or 2, the image of the product

$$H^0(C, N_{C/X} - L_i) \otimes H^0(C, K_C + L_i) \to H^0(C, K_C + N_{C/X})$$

contains the subspace $H^0(C, K_C + N_{C/X} - B) \subset H^0(C, K_C + N_{C/X})$ of sections that vanish on B, by the previous paragraph.

As a result, the lemma is proved if the product

$$H^0(C, K_C + N_{C/X} - B) \otimes H^0(C, K_C) \to H^0(C, 2K_C + N_{C/X} - B) \subset H^0(C, 2K_C + N_{C/X}) \xrightarrow{\beta} k$$

is nonzero. Indeed, if that holds, then at least one of the two order-p line bundles L_1 or L_2 will have class in $H^0(C, K_C)$ whose product with $H^0(C, K_C + N_{C/X} - B)$ has nonzero image under β, which implies the statement (*) above.

By Castelnuovo's theorem [1, p. 151], the product map

$$H^0(C, K_C + N_{C/X} - B) \otimes H^0(C, K_C) \to H^0(C, 2K_C + N_{C/X} - B)$$

is surjective, because C has genus g at least 2 and $K_C + N_{C/X} - B$ has degree at least $2g + 1$ (in fact, it has degree 10, which is at least $2g + 1 = 5$). So it remains only to show that β is nonzero on the subspace $H^0(C, 2K_C + N_{C/X} - B)$ of $H^0(C, 2K_C + N_{C/X})$ for every effective divisor B of degree 4. This follows by Serre duality from the property of β in $H^1(C, TC - N_{C/X})$ proved at the beginning of this proof: the image of β in $H^1(C, TC - N_{C/X} + B)$ is nonzero for every effective divisor B of degree 4 on C. QED (Lemma 6.2).

Lemma 6.4 *Let (C, L) be a general pair over $\overline{\mathbf{F}}_p$ with C a smooth ordinary curve of genus 2 and L a line bundle of order p on C. Then the Frobenius map $F^* : H^1(C, L) \to H^1(C, pL) \cong H^1(C, O)$ is injective. (Here $H^1(C, L)$ has dimension 1 and $H^1(C, O)$ has dimension 2.)*

Note that the moduli space of pairs (C, L) over $\overline{\mathbf{F}}_p$ with C ordinary of genus 2 and L of order p is irreducible, by the irreducibility of the moduli space of curves of genus 2 together with the theorem that the geometric monodromy homomorphism for ordinary curves (on the group $(\mathbf{Z}/p)^2$ of line bundles killed by p) maps onto $GL_2(\mathbf{Z}/p)$ (Faltings-Chai [6, Prop. V.7.1], Ekedahl [5]). As a result, it makes sense to talk about a general pair (C, L), meaning the pairs outside some proper closed subset of the moduli space.

Proof. Let C_0 be the union of two ordinary elliptic curves E_1 and E_2 identified at the origin. Since E_1 is ordinary, there is a line bundle L on C_0 that has order p on E_1 and is trivial on E_2. Then $H^1(C_0, O) \cong H^1(E_1, O) \oplus H^1(E_2, O) \cong k^2$. Also, $H^1(C_0, L) \cong H^1(E_1, L) \oplus H^1(E_2, O) \cong k$. So the Frobenius map $F^* : H^1(C_0, L) \to H^1(C_0, pL) \cong H^1(C, O)$ is the Frobenius map $H^1(E_2, O) \to H^1(E_2, O) \subset H^1(E_1, O) \oplus H^1(E_2, O)$. Since E_2 is ordinary, this Frobenius map is nonzero, hence injective.

We can deform C_0 to smooth ordinary curves of genus 2. In this deformation, the Jacobians form a smooth family of ordinary abelian surfaces. Therefore the line bundle L of order p on C_0 can be deformed to a line bundle of order p over the smooth curves (over an etale open subset of the parameter space). It follows that $F^* : H^1(C, L) \to H^1(C, pL)$ is injective for some pair (C, L) with C smooth ordinary of genus 2 and L of order p, hence for general such pairs. QED

We can now give the first examples of nef but not semi-ample line bundles on smooth projective varieties over $\overline{\mathbf{F}}_p$. We know that a general smooth curve C of genus 2 is ordinary. It follows from Lemma 6.4 that, for a general ordinary smooth curve C of genus 2, every line bundle L of order p on C has injective Frobenius map $F^* : H^1(C, L) \to H^1(C, pL) \cong H^1(C, O)$. Let C be an ordinary smooth curve of genus 2 over $\overline{\mathbf{F}}_p$ that has this injectivity property.

We can imbed this curve, like any smooth curve C of genus 2, as a curve of bidegree $(2, 3)$ in $\mathbf{P}^1 \times \mathbf{P}^1$. Explicitly: use the line bundles K_C and any line bundle A of degree 3 that is not of the form $K_C + p$ for any point p in C. The line bundles

K_C and A are both basepoint-free and have $h^0 = 2$, and so they give two morphisms $C \to \mathbf{P}^1$, hence a morphism $C \to \mathbf{P}^1 \times \mathbf{P}^1$ of bidegree $(2, 3)$. It is straightforward to check from the assumption on A that $C \to \mathbf{P}^1 \times \mathbf{P}^1$ is an embedding.

By Lemma 6.2, there is a divisor D of degree 12 on C over $\overline{\mathbf{F}}_p$ such that blowing up $\mathbf{P}^1 \times \mathbf{P}^1$ at the divisor D gives a surface M with the following property. The line bundle $L := O(C)$ on M has order p when restricted to C, while pL is nontrivial on $2C$. (Here C denotes the proper transform in M of the curve with the same name in $\mathbf{P}^1 \times \mathbf{P}^1$.)

We have arranged for all the hypotheses of Theorem 5.1. Therefore $L = O(C)$ is not semi-ample on M, and in fact no multiple of C moves on M. QED

7 Nef and big but not semi-ample

To conclude, it is easy to use our examples in dimension 2 to produce a similar example in dimension 3, but now involving a nef *and big* line bundle. In the simplest example, the 3-fold is rational. By the argument in section 4, this gives examples on many other varieties, in particular on 3-folds of general type.

Theorem 7.1 *For any prime number p, there is a nef and big line bundle L on a smooth projective 3-fold W over $\overline{\mathbf{F}}_p$ which is not semi-ample.*

Equivalently, by Theorem 1.2, the ring $R(W, L) = \oplus_{a \geq 0} H^0(W, aL)$ is not finitely generated over $\overline{\mathbf{F}}_p$. There is no example as in Theorem 7.1 in dimension 2, by Artin [2, proof of Theorem 2.9(B)] or Keel [12].

Proof. By Theorem 6.1, for every prime number p, there is a smooth projective surface X over $k = \overline{\mathbf{F}}_p$ and a nef line bundle L_1 on X which is not semi-ample. Equivalently, as mentioned in section 1, if we let L_2 be an ample line bundle on X, then the ring $R(X, L_1, L_2) := \oplus_{a,b \geq 0} H^0(X, aL_1 + bL_2)$ is not finitely generated over k. Since L_2 and $L_1 + L_2$ are ample, this ring has Iitaka dimension 3, meaning that the subspaces of total degree d grow at least like a positive constant times d^3.

Let W be the projective bundle $P(L_1 \oplus L_2)$ of hyperplanes in $L_1 \oplus L_2$. The line bundle $O(1)$ on the $\mathbf{P}^1$-bundle $\pi : W \to X$ is easily checked to be nef, since L_1 and L_2 are nef. We have $\pi_* O(1) = L_1 \oplus L_2$ and more generally $\pi_* O(d) = S^d(L_1 \oplus L_2) = \oplus_{i=0}^{d} L_1^{\otimes i} \oplus L_2^{\otimes d-i}$. Therefore

$$R(W, O(1)) = R(X, L_1, L_2).$$

So the ring $R(W, O(1))$ has Iitaka dimension 3 but is not finitely generated. Therefore the nef line bundle $O(1)$ on the 3-fold W is big but not semi-ample. QED

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