ENDOMORPHISMS OF VARIETIES AND BOTT VANISHING

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ABSTRACT. We show that a projective variety with an int-amplified endomorphism of degree invertible in the base field satisfies Bott vanishing. This is a new way to analyze which varieties have nontrivial endomorphisms. In particular, we extend some classification results on varieties admitting endomorphisms (for Fano threefolds of Picard number one and several other cases) to any characteristic. The classification results in characteristic zero are due to Amerik–Rovinsky–Van de Ven, Hwang–Mok, Paranjape–Srinivas, Beauville, and Shao–Zhong. Our method also bounds the degree of morphisms into a given variety. Finally, we relate endomorphisms to global F-regularity.

1. INTRODUCTION

There is a long-standing conjecture about smooth Fano varieties admitting non-invertible surjective endomorphisms.

Conjecture 1.1. *Let X be a smooth Fano variety of Picard number 1 over an algebraically closed field of characteristic zero. Suppose that X admits a non-invertible surjective endomorphism. Then X is isomorphic to projective space.*

Conjecture 1.1 has been proved when

(1) dim $X = 3$ [Ame97, ARVdV99, HM03],
(2) dim $X = 4$ and X has Fano index greater than 1 [SZ24],
(3) X is a hypersurface [PS89, Bea01], or
(4) X is a homogeneous space [PS89].

The aim of this paper is to give a new approach to this problem and to generalize cases (1), (2), and (3) above to arbitrary characteristic.

Theorem A. *Let X be a smooth projective variety over an algebraically closed field k. Assume that X admits an endomorphism whose degree is greater than 1 and invertible in k. Suppose that one of the following holds.*

(1) *X is a smooth Fano threefold of Picard number 1.*
(2) *X is a smooth Fano fourfold of Picard number 1 and Fano index greater than 1.*
(3) *X is a hypersurface of dimension at least 3.*

Then X is isomorphic to projective space.

Our method also gives information on morphisms other than endomorphisms. The following result was known in characteristic zero in cases (1) and (2) [Ame97], [ARVdV99, Theorem 0.2], [HM03, Theorem 2], [SZ24, Theorem 1.5], and for quadrics in characteristic zero in case (3) [Ame07, Theorem]. Our proof is short and valid in arbitrary characteristic.

Theorem B. *Let X be one of the varieties in Theorem A. Let Y be a smooth projective variety over k of the same dimension that also has Picard number 1. If X is not isomorphic to projective space, then there is an upper bound on the degrees of all morphisms $Y \to X$ that have degree invertible in k.*

The following assertion is a key ingredient for Theorem A. An endomorphism $f \colon X \to X$ is said to be *int-amplified* if there is an ample Cartier divisor H on X such that $f^*H - H$ is ample [Men20, MZ20].

Theorem C. *Let X be a normal projective variety over a perfect field k. Suppose that X admits an int-amplified endomorphism whose degree is invertible in k. Then X satisfies Bott vanishing for ample Weil divisors. That is,*

$$H^i(X, \Omega_X^{[j]}(A)) = 0$$

for every $i > 0$, $j \geq 0$, and A an ample Weil divisor.

Remark 1.2. The assumption "int-amplified" is weaker than some related conditions on endomorphisms, such as *polarized*, meaning that there is an ample Cartier divisor H with $f^*H \sim qH$ for some integer $q \geq 2$. For example, the endomorphism $f(x, y) = (x^2, y^3)$ of $\mathbb{P}^1 \times \mathbb{P}^1$ is int-amplified, but no positive iterate of f is polarized. On the other hand, for a variety with Picard group $\mathbb{Z}$, every endomorphism of degree greater than 1 is polarized and hence int-amplified.

Remark 1.3. A smooth Fano variety that satisfies Bott vanishing is rigid, since $H^1(X, TX) = H^1(X, \Omega_X^{d-1}(-K_X)) = 0$, where d is the dimension of X. So Theorem C implies that only finitely many smooth complex Fano varieties in each dimension admit an int-amplified endomorphism.

Remark 1.4. In proving Theorem C for singular varieties, we develop some interesting tools. In particular, we prove the finiteness of flat cohomology $H^1(X, \mu_p)$ for smooth varieties X over an algebraically closed field of characteristic p (Lemma 2.1).

Finally, we show that a Fano variety with a suitable endomorphism is well-behaved in characteristic p:

Theorem D. *Let X be a Fano variety over a perfect field k of characteristic $p > 0$. Suppose that X admits an int-amplified endomorphism of degree prime to p. If X is strongly F-regular (for example, smooth), then it is globally F-regular.*

It is known that the mod p reductions of a klt Fano variety in characteristic zero are globally F-regular for sufficiently large primes p ([SS10, Theorem 1.2]). The point of Theorem D is that it holds even if p is small. In this respect, Fano varieties with a suitably nontrivial endomorphism behave well, somewhat like toric varieties. For

example, Petrov showed that the Hodge spectral sequence degenerates for all smooth projective varieties that are globally F-split (which follows from globally F-regular) [Bha22, Corollary 2.7.6].

For hypersurfaces of dimension at least 3 and degree at least 2, it is straightforward to see that Bott vanishing fails. This implies Theorem A(3) by Theorem C.

The proof of Theorem A is similar for cases (1) and (2). To describe case (1): we show that projective space is the only smooth Fano threefold of Picard number 1 that satisfies Bott vanishing (see also Remark 1.5). In characteristic zero, this is an easy consequence of the classification [IP99]. So assume that the characteristic p is greater than 0. Since we do not have such a complete classification in this case, we lift X to characteristic zero. By Theorem C, we can take a lift $\widetilde{X}_{\overline{K}}$ of X, which is a smooth Fano threefold of Picard number 1 over an algebraically closed field $\overline{K}$ of characteristic zero. However, lifting endomorphisms is difficult in general, and therefore we prove that $\widetilde{X}_{\overline{K}}$ satisfies Bott vanishing instead. Then $\widetilde{X}_{\overline{K}} \cong \mathbb{P}^3_{\overline{K}}$ by the argument in characteristic zero. Finally, observing that the Fano indices are preserved by lifting, we conclude that $X \cong \mathbb{P}^3_k$.

Remark 1.5. The paper [Tot24] determines which smooth Fano threefolds in characteristic zero satisfy Bott vanishing [Tot24, Theorem 0.1]. For smooth Fano threefolds in positive characteristic that are constructed in the same way as smooth Fano threefolds in characteristic zero, that paper also determines which ones satisfy Bott vanishing. After this paper appeared on the arXiv, Tanaka showed that the classification of smooth Fano threefolds takes essentially the same form in any characteristic [Tan23b, Theorem 1.1], [Tan23a, Theorem 1.1]. Independent of Tanaka's results, we show unconditionally that projective space is the only smooth Fano threefold of Picard number 1 in any characteristic that satisfies Bott vanishing (Proposition 3.8).

1.1. **Related results.**

1.1.1. *Two-dimensional case in positive characteristic.* Nakayama showed that a smooth projective rational surface in characteristic p that admits an endomorphism whose degree is greater than 1 and prime to p must be toric [Nak10, Proposition 4.4]. On the other hand, he found a smooth rational surface that admits a separable polarized endomorphism but is not toric [Nak10, Example 4.5].

1.1.2. *Failure of Bott vanishing for separable polarized endomorphisms.* Answering a question in the first version of this paper, we give an example to show that Theorem C fails (that is, Bott vanishing fails) if the assumption that the endomorphism f has degree invertible in k is weakened to the assumption that f is separable (Proposition 5.1). We do not know whether Theorems A and B hold for f separable rather than for f of degree invertible in k.

1.1.3. *Three-dimensional case in arbitrary characteristic.* Normal projective $\mathbb{Q}$-Gorenstein threefolds that admit polarized endomorphisms in arbitrary characteristic were studied in detail by Cascini–Meng–Zhang [CMZ20, Theorem 1.8]. In particular, they proved that a smooth rationally chain connected threefold in characteristic p

admitting a polarized endomorphism f has an f-equivariant minimal model program if $p > 5$ and the degree of the Galois closure f^{Gal} of f is prime to p.

1.1.4. *Fano threefolds of arbitrary Picard number in characteristic zero.* Meng–Zhang–Zhong [MZZ22] proved that a smooth Fano threefold over an algebraically closed field of characteristic zero that admits an int-amplified endomorphism is toric. Therefore, it is natural to ask the following question. After the first version of this paper, a positive answer was given in [Tot23, Theorem 6.1].

Question 1.6. Let X be a smooth Fano threefold over an algebraically closed field of characteristic $p > 0$. Suppose that X admits an int-amplified endomorphism whose degree is prime to p. Is X toric?

1.2. **Notation and terminology.** Unless otherwise mentioned, k is an algebraically closed field of characteristic $p \geq 0$ and a variety is defined over k.

A morphism of varieties $f \colon Y \to X$ over k is *separable* if it is dominant and $k(Y)$ is a separable field extension of $k(X)$. Equivalently (for k algebraically closed), the derivative of f is surjective at some smooth point of Y.

For a normal variety X over a field k and $i \geq 0$, we write Ω^i_X for $\Omega^i_{X/k}$. The sheaf of *reflexive differentials* $\Omega^{[i]}_X$ is the double dual $(\Omega^i_X)^{**}$. More generally, for a Weil divisor D on X, we write $\Omega^{[i]}_X(D)$ for the reflexive sheaf $(\Omega^i_X \otimes \mathcal{O}_X(D))^{**}$. If X is smooth over k, then $\mathcal{O}_X(D)$ is a line bundle and $\Omega^{[i]}_X(D)$ is just the tensor product $\Omega^i_X \otimes \mathcal{O}_X(D)$.

2. Bott vanishing and endomorphisms

In this section, we prove Theorem C, relating Bott vanishing with endomorphisms. We also prove a general relation between Bott vanishing and morphisms into a given variety, not just endomorphisms (Proposition 2.7).

2.1. **Finiteness of flat cohomology.** The following lemma may be known, but we could not find a reference in this generality.

Lemma 2.1. *Let X be a smooth variety over an algebraically closed field k, and let s be a positive integer. Then the flat cohomology group $H^1(X, \mu_s)$ is finite.*

Proof. The problem reduces to the case where s is prime. If s is invertible in k, then this finiteness holds for cohomology in all degrees [Mil80, Corollary VI.5.5]. So we can assume that k has characteristic $p > 0$ and $s = p$. (The result here is special to H^1. Indeed, a supersingular K3 surface X over k has $H^2(X, \mu_p)$ infinite, containing the additive group k [Art74, Proposition 4.2].) It is straightforward to see that $H^1(X, \mu_p)$ injects into $H^1(k(X), \mu_p) = k(X)^*/(k(X)^*)^p$ [Kel22, Lemma 3.9].

Let $\overline{X}$ be a normal compactification of X. By de Jong, there is a separable alteration $f \colon \overline{Y} \to \overline{X}$ [dJ96, Theorem 4.1]. That is, f is generically étale and $\overline{Y}$ is a smooth projective variety over k. Let Y be the inverse image of X in $\overline{Y}$. Here $k(Y)$ is a finite separable extension of $k(X)$, and so $k(X)^*/(k(X)^*)^p$ injects into $k(Y)^*/(k(Y)^*)^p$. By the previous paragraph, it follows that $H^1(X, \mu_p)$ injects into $H^1(Y, \mu_p)$.

So it suffices to show that $H^1(Y, \mu_p)$ is finite. Consider the Kummer sequence $\mathcal{O}(Y)^*/(\mathcal{O}(Y)^*)^p \to H^1(Y, \mu_p) \to \mathrm{Pic}(Y)[p]$. The group of units $\mathcal{O}(Y)^*$ is an extension of a finitely generated abelian group by k^* [Kah06, Lemme 1], and so $\mathcal{O}(Y)^*/(\mathcal{O}(Y)^*)^p$ is finite. So it suffices to show that $\mathrm{Pic}(Y)[p]$ is finite.

Since $\overline{Y}$ is smooth, we have $\mathrm{Pic}(Y) = \mathrm{Pic}(\overline{Y})/M$, where M is the subgroup generated by the codimension-1 subvarieties of $\overline{Y}$ contained in $\overline{Y} - Y$. In particular, the abelian group M is finitely generated. Applying the snake lemma to the map of exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & \mathrm{Pic}(\overline{Y}) & \longrightarrow & \mathrm{Pic}(Y) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \\
0 & \longrightarrow & M & \longrightarrow & \mathrm{Pic}(\overline{Y}) & \longrightarrow & \mathrm{Pic}(Y) & \longrightarrow & 0,
\end{array}
$$

we obtain an exact sequence $\mathrm{Pic}(\overline{Y})[p] \to \mathrm{Pic}(Y)[p] \to M/p$. So $\mathrm{Pic}(Y)[p]$ is finite if $\mathrm{Pic}(\overline{Y})[p]$ is finite. That is immediate from the structure of the Picard group of a smooth projective variety: $\mathrm{Pic}(\overline{Y})$ is an extension of a finitely generated abelian group $NS(\overline{Y})$ by the k-points of an abelian variety [BGI71, Théorème XIII.5.1]. Lemma 2.1 is proved. □

2.2. **Bott vanishing.** In this subsection, we prove Theorem C. We are generalizing Fujino's proof of Bott vanishing for toric varieties, based on the existence of suitable endomorphisms, beyond the toric setting.

Definition 2.2. Let X be a smooth projective variety over a field. We say that X satisfies *Bott vanishing* if we have

$$H^i(X, \Omega^j_X(A)) = 0$$

for every $i > 0$, $j \geq 0$, and A an ample Cartier divisor.

For singular varieties, we consider the following strong version of Bott vanishing. A Weil divisor (with integer coefficients) is called *ample* if some positive multiple is an ample Cartier divisor. Likewise for *nef*.

Definition 2.3. Let X be a normal projective variety over a field. We say that X satisfies *Bott vanishing for ample Weil divisors* if we have

$$H^i(X, \Omega^{[j]}_X(A)) = 0$$

for every $i > 0$, $j \geq 0$, and A an ample Weil divisor.

Remark 2.4. All projective toric varieties satisfy Bott vanishing for ample Weil divisors, by Fujino [Fuj07, Proposition 3.2].

Proof of Theorem C. Since k is perfect, the normal variety X over k is geometrically normal [SPA23, Tag 038O]. Replacing k with its algebraic closure, we may assume that k is algebraically closed. For clarity, we first prove the theorem for X smooth. (That is enough for the applications in this paper.) So let X be a smooth projective variety over an algebraically closed field k with an endomorphism f and an ample

Cartier divisor H on X such that $f^*H - H$ is ample. In particular, f^*H is ample, and so f does not contract any curves. Therefore, $f\colon X \to X$ is finite. We assume that the degree of f is invertible in k.

Let A be any ample Cartier divisor on X. Fix $i > 0$ and $j \geq 0$. We want to show that
$$H^i(X, \Omega_X^j(A)) = 0.$$
We will use Fujita's vanishing theorem [Fuj83, Theorem 1]:

Theorem 2.5. *Let X be a projective scheme over a field, H an ample Cartier divisor on X, and E a coherent sheaf on X. Then there is a positive integer m such that $H^i(X, E \otimes \mathcal{O}_X(mH + D)) = 0$ for every $i > 0$ and every nef Cartier divisor D on X.*

Since f is finite and X is smooth over k, there is a *trace map* $\tau_f\colon f_*\Omega_X^j \to \Omega_X^j$ such that the composition
$$\Omega_X^j \xrightarrow{f^*} f_*\Omega_X^j \xrightarrow{\tau_f} \Omega_X^j$$
is multiplication by $\deg(f)$, by Garel and Kunz [Gar84], [Kun86, section 16], [SPA23, Tag 0FLC]. Thus $\frac{1}{\deg(f)}\tau_f$ gives a splitting of the pullback $f^*\colon \Omega_X^j \hookrightarrow f_*\Omega_X^j$. Taking the pushforward by f, we obtain a split injective map $f_*\Omega_X^j \hookrightarrow (f^2)_*\Omega_X^j$, and thus a split injective map $\Omega_X^j \hookrightarrow (f^2)_*\Omega_X^j$. Repeating this procedure, for every positive integer e, $(f^e)^*\colon \Omega_X^j \hookrightarrow (f^e)_*\Omega_X^j$ splits. Tensoring with $\mathcal{O}_X(A)$, we have a split injective map
$$\Omega_X^j(A) \to (f^e)_*(\Omega_X^j((f^e)^*A)).$$
Taking cohomology (and using that f^e is finite), we have a split injective map
$$H^i(X, \Omega_X^j(A)) \hookrightarrow H^i(X, \Omega_X^j((f^e)^*A)).$$
So it suffices to find an $e \geq 1$ such that the right hand side is zero. Let m be a positive integer associated to the given ample Cartier divisor H (with $f^*H - H$ ample) and the coherent sheaf $E = \Omega_X^i$, in Fujita vanishing (Theorem 2.5). Then it suffices to find an $e \geq 1$ such that $(f^e)^*A - mH$ is nef.

Since $f^*H - H$ is ample, there is a rational number $c > 1$ such that $f^*H - cH$ is ample. Here f^* takes nef divisors to nef divisors. So, for every $e \geq 1$, $(f^e)^*H - c^eH$ is nef.

Since A is ample, there is a rational number $u > 0$ such that $A - uH$ is ample. Using again that f^* takes nef divisors to nef divisors, we find that for every $e \geq 1$, $(f^e)^*A - u(f^e)^*H$ is nef. It follows that $(f^e)^*A - uc^eH$ is nef. There is a positive integer e such that $uc^e \geq m$. Then $(f^e)^*A - mH$ is nef, as we want. Bott vanishing is proved.

More generally, assume that X is normal (rather than smooth). Since f is a finite morphism between normal varieties, reflexive differentials $\Omega_X^{[j]} = (\Omega_X^j)^{**}$ pull back under f. (We can pull differential forms back outside $X^{\mathrm{sing}} \cup f(X^{\mathrm{sing}})$, and that gives a pullback map on reflexive differentials since the complement has codimension at least 2.) That is, we have a pullback map $\Omega_X^{[j]} \to f_*\Omega_X^{[j]}$. Likewise, the trace map $\tau_f\colon f_*\Omega_X^{[j]} \to \Omega_X^{[j]}$ is defined outside $X^{\mathrm{sing}} \cup f(X^{\mathrm{sing}})$, so it extends to all of X since

the sheaf $\Omega_X^{[j]}$ is reflexive. Given this, the proof above works without change for an ample Cartier divisor A.

Finally, to prove the full theorem, assume that X is normal and A is an ample Weil divisor. For this, we need a version of Fujita vanishing for $\mathbb{Q}$-Cartier Weil divisors:

Lemma 2.6. *Let X be a normal projective variety over an algebraically closed field k, H an ample Cartier divisor on X, E a reflexive sheaf on X, and s a positive integer. Then there is a positive integer m such that $H^i(X, E(mH + D)) = 0$ for every $i > 0$ and every nef Weil divisor D such that sD is Cartier.*

Proof. Since X is a normal projective variety over k, the Picard group $\mathrm{Pic}(X)$ is an extension of a finitely generated abelian group by the k-points of an abelian variety [BGI71, Théorème XIII.5.1], [Kle05, Theorem 5.4]. Therefore, $\mathrm{Pic}(X)/s$ is finite. Let U be the smooth locus of X, so that the divisor class group $\mathrm{Cl}(X)$ is isomorphic to $\mathrm{Pic}(U)$. By Lemma 2.1, $H^1(U, \mu_s)$ is finite. By the Kummer sequence $H^1(U, \mu_s) \to H^1(U, G_m) \xrightarrow{s} H^1(U, G_m)$, the s-torsion subgroup $\mathrm{Cl}(X)[s] = \mathrm{Pic}(U)[s]$ is also finite. By tensoring the exact sequence $0 \to \mathrm{Pic}(X) \to \mathrm{Cl}(X) \to \mathrm{Cl}(X)/\mathrm{Pic}(X) \to 0$ over $\mathbb{Z}$ with $\mathbb{Z}/s$, we have an exact sequence $\mathrm{Cl}(X)[s] \to (\mathrm{Cl}(X)/\mathrm{Pic}(X))[s] \to \mathrm{Pic}(X)/s$. So $(\mathrm{Cl}(X)/\mathrm{Pic}(X))[s]$ is finite.

Let $D_1, \ldots, D_r$ be Weil divisors with sD_j Cartier that represent every element of the group $(\mathrm{Cl}(X)/\mathrm{Pic}(X))[s]$. By subtracting a suitable multiple of H from each D_j, we can assume that $-D_j$ is nef for each j. Then, for every nef Weil divisor D on X with sD Cartier, there is a $1 \le j \le r$ such that $D - D_j$ is Cartier. Apply Fujita's theorem (Theorem 2.5) to the coherent sheaf $\oplus_{j=1}^r E(D_j)$. (By definition, $E(D_j)$ means the reflexive sheaf $(E \otimes \mathcal{O}_X(D_j))^{**}$.) This gives that there is a positive integer m such that $H^i(X, E(mH + D_j + N)) = 0$ for every $i > 0$, every $1 \le j \le r$, and every nef Cartier divisor N. Since $-D_j$ is nef for each j, it follows that for every nef Weil divisor D with sD Cartier, we have $H^i(X, E(mH + D)) = 0$ for every $i > 0$. Lemma 2.6 is proved. $\square$

We now complete the proof of Theorem C for X normal over an algebraically closed field k and an ample Weil divisor A. This follows by the argument above (where A is an ample Cartier divisor), using Lemma 2.6 in place of Fujita's theorem. There is a positive integer s such that sA is Cartier. Since f is a finite morphism between normal varieties, $(f^e)^*A$ is a Weil divisor (with integer coefficients) for every positive integer e, and $s(f^e)^*A$ is Cartier. This is what we need in order to apply Lemma 2.6. Theorem C is proved. $\square$

2.3. Bounding the degree of morphisms.

Proposition 2.7. *Let X and Y be normal projective varieties of the same dimension over a field k, and suppose that both have Picard number 1. If there are morphisms from Y to X with arbitrarily large degree such that the degree is invertible in k, then X must satisfy Bott vanishing for ample Weil divisors.*

Proof. Let $f\colon Y \to X$ be a morphism whose degree is invertible in k. Let H be an ample Cartier divisor on X. Then f^*H has positive degree on some curve, hence is ample since Y has Picard number 1. It follows that $f\colon Y \to X$ is finite.

Let A be an ample Weil divisor on X, and fix $i > 0$ and $j \geq 0$. We need to prove that

$$H^i(X, \Omega_X^{[j]}(A)) = 0.$$

Since $\deg(f)$ is invertible in k, we have a split injection

$$\Omega_X^{[j]}(A) \hookrightarrow f_*(\Omega_Y^{[j]}(f^*A)),$$

as in the proof of Theorem C. Taking cohomology (and using that f is finite), we have a split injection

$$H^i(X, \Omega_X^{[j]}(A)) \hookrightarrow H^i(Y, \Omega_Y^j(f^*A)).$$

If there are morphisms $f\colon Y \to X$ with arbitrarily large degree such that the degree is invertible in k, then f^*A becomes arbitrarily large in the ample cone of Y (here just one ray). So Fujita vanishing for Weil divisors (Lemma 2.6) gives that, for f of sufficiently large degree, $H^i(Y, \Omega_Y^{[j]}(f^*A)) = 0$. By the previous paragraph, it follows that $H^i(X, \Omega_X^{[j]}(A)) = 0$. Bott vanishing is proved. $\qquad\square$

3. Bott vanishing for specific classes of varieties

3.1. **Hypersurfaces.** In this subsection, we prove that projective space is the only smooth hypersurface of dimension at least 3 that satisfies Bott vanishing.

Lemma 3.1. *Let $X \subset \mathbb{P}^{d+1}$ be a smooth hypersurface of degree $d + n$ over a field. Suppose that $d > 1$ and $n > 0$. Then $H^{d-1}(X, \Omega_X^1(n)) \neq 0$, and in particular, X does not satisfy Bott vanishing.*

Proof. By adjunction, the dualizing sheaf ω_X is isomorphic to $\mathcal{O}_X(n-2)$. So Serre duality gives that $H^d(X, \mathcal{O}_X(-d)) \cong H^0(X, \mathcal{O}_X(d+n-2))^* \neq 0$. By the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{d+1}} \to \mathcal{O}_{\mathbb{P}^{d+1}}(1)^{\oplus d+2} \to T\mathbb{P}^{d+1} \to 0$$

we have $H^0(\mathbb{P}^{d+1}, T\mathbb{P}^{d+1}(-2)) = 0$. By the exact sequence

$$0 \to \Omega_{\mathbb{P}^{d+1}}^1(-d) \to \Omega_{\mathbb{P}^{d+1}}^1(n) \to \Omega_{\mathbb{P}^{d+1}}^1(n)|_X \to 0,$$

we have $H^d(X, \Omega_{\mathbb{P}^{d+1}}^1(n)|_X) = 0$. Here we used that $H^{d+1}(\mathbb{P}^{d+1}, \Omega_{\mathbb{P}^{d+1}}^1(-d)) \cong H^0(\mathbb{P}^{d+1}, T\mathbb{P}^{d+1}(-2))^* = 0$ and Bott vanishing on $\mathbb{P}^{d+1}$. By the exact sequence

$$0 \to \mathcal{O}_X(-d) \to \Omega_{\mathbb{P}^{d+1}}^1(n)|_X \to \Omega_X^1(n) \to 0,$$

we have $H^{d-1}(X, \Omega_X^1(n)) \neq 0$, as desired. $\qquad\square$

Proposition 3.2. *Let $X \subset \mathbb{P}^{d+1}$ be a smooth hypersurface of dimension at least 2 over an algebraically closed field. If X satisfies Bott vanishing, then $X \cong \mathbb{P}^d$ or X is the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. A Fano variety that satisfies Bott vanishing is rigid, since $H^1(X, TX) = H^1(X, \Omega_X^{d-1}(-K_X)) = 0$, where $d = \dim(X)$. That excludes all Fano hypersurfaces of degree at least 3 [Kod86, equation 5.21]. Bott vanishing also fails for quadrics of dimension at least 3 [AWZ21, Example 3.2.6]. Finally, Lemma 3.1 shows that Bott vanishing fails for all non-Fano hypersurfaces of dimension at least 2. (This is mostly relevant for the Calabi-Yau case. Indeed, Bott vanishing fails for all varieties of positive dimension d with ample canonical class, since $H^d(X, \omega_X) = k \neq 0$.) $\square$

3.2. **Fano threefolds.** In this subsection, we prove that projective space is the only Fano threefold of Picard number 1 that satisfies Bott vanishing, in any characteristic. Throughout this subsection, we use the following convention.

Convention 3.3. Let k be an algebraically closed field of positive characteristic. We denote $W(k)$ the ring of Witt vectors and K the field of fractions of $W(k)$. For a proper scheme X over k, we say that a scheme $\widetilde{X}$ over $W(k)$ is a *lift of X* if $\widetilde{X} \otimes_{W(k)} k \cong X$ and $\widetilde{X}$ is flat and proper over $W(k)$. For a lift $\widetilde{X}$ and a Cartier divisor $\widetilde{A}$ on $\widetilde{X}$, we denote the geometric generic fiber of $\widetilde{X} \to \operatorname{Spec} W(k)$ by $\widetilde{X}_{\overline{K}}$ and the pullback of $\widetilde{A}$ to $\widetilde{X}_{\overline{K}}$ by $\widetilde{A}_{\overline{K}}$.

Definition 3.4. Let X be a smooth Fano variety over an algebraically closed field. The *Fano index $r(X) \in \mathbb{Z}_{>0}$* of X is the largest integer $n \in \mathbb{Z}_{>0}$ such that $-K_X \sim nA$ for some Cartier divisor A.

Lemma 3.5. *Let X be a smooth Fano variety with $d := \dim X$. Suppose that X satisfies Kodaira vanishing. If $r(X) \geq d + 1$, then $X \cong \mathbb{P}^d$.*

Proof. The proof of [Meg98, Proposition 4] works in any dimension, since we assume that X satisfies Kodaira vanishing. $\square$

Proposition 3.6. *Let X be a smooth Fano variety over an algebraically closed field k of positive characteristic. Suppose there exists a lift $\widetilde{X}$ over $W(k)$ of X. In addition, assume that $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. Then the following statements hold.*

(1) *The specialization map $\operatorname{sp} \colon \operatorname{Pic}(\widetilde{X}_{\overline{K}}) \to \operatorname{Pic}(X)$ is an isomorphism of abelian groups.*
(2) *$r(X) = r(\widetilde{X}_{\overline{K}})$.*
(3) *Suppose that the Picard number of X is 1. If X satisfies Bott vanishing, then so does $\widetilde{X}_{\overline{K}}$.*

Remark 3.7. For a smooth Fano threefold X, Shepherd-Barron [SB97, Corollary 1.5] proved that $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ (see also [Kaw21, Corollary 3.7] for the case where $p = 2$ or 3). Also, for a smooth Fano variety X of any dimension that satisfies Bott vanishing (hence Kodaira vanishing), we have $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$. See [GJ18, Theorem 1.1, Proposition 6.3] for what is known about the Fano index in families without assuming that $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$.

Proof. First, we prove (1). Since smoothness and ampleness are open properties in a flat proper family, the geometric generic fiber $\widetilde{X}_{\overline{K}}$ is a smooth Fano variety

and $\text{Pic}(\widetilde{X}_{\overline{K}}) \cong \text{NS}(\widetilde{X}_{\overline{K}})$ is a free $\mathbb{Z}$-module, where $\text{NS}(X)$ denotes the Néron-Severi group. Let $\text{sp}\colon \text{Pic}(\widetilde{X}_{\overline{K}}) \to \text{Pic}(X)$ be the specialization map (see the proof of [MP12, Proposition 3.3] for the construction). Since $H^1(X, \mathcal{O}_X) = 0$, we have $\text{Pic}(X) = \text{NS}(X)$ by [FGI$^+$05, Theorem 9.5.11]. The Picard group of the smooth Fano variety $\widetilde{X}_{\overline{K}}$ in characteristic zero is torsion-free; for a quick proof, see [FS20, Introduction]. Next, [MP12, Proposition 3.6] gives that the specialization $NS(\widetilde{X}_{\overline{K}}) \to NS(X)$ is injective. Since $\text{Pic}(X_{\overline{K}}) = NS(X_{\overline{K}})$ and $\text{Pic}(X) = NS(X)$ in our case, it follows that $\text{sp}\colon \text{Pic}(\widetilde{X}_{\overline{K}}) \to \text{Pic}(X)$ is injective. Since $H^2(X, \mathcal{O}_X) = 0$, the specialization map sp is also surjective [FGI$^+$05, Corollary 8.5.6]. Thus (1) holds.

Next, we prove (2). By the definition of $r(\widetilde{X}_{\overline{K}})$, we can take an ample Cartier divisor $\overline{A}$ on $\widetilde{X}_{\overline{K}}$ such that $-K_{\widetilde{X}_{\overline{K}}} \sim r(\widetilde{X}_{\overline{K}})\overline{A}$. Then we have

$$-K_X \sim \text{sp}(-K_{\widetilde{X}_{\overline{K}}}) \sim \text{sp}(r(\widetilde{X}_{\overline{K}})\overline{A}) = r(\widetilde{X}_{\overline{K}})\text{sp}(\overline{A}),$$

which shows that $r(\widetilde{X}_{\overline{K}}) \leq r(X)$.

By the definition of $r(X)$, we can take an ample Cartier divisor A on X such that $-K_X \sim r(X)A$. Let $\widetilde{A} \in \text{Pic}(\widetilde{X})$ be a lift of A. Then we obtain $-K_{\widetilde{X}_{\overline{K}}} \sim r(X)\widetilde{A}_{\overline{K}}$, which shows that $r(X) \leq r(\widetilde{X}_{\overline{K}})$. Thus, (2) holds.

Finally, we prove (3). Take an ample Cartier divisor $\overline{A}$ on $\widetilde{X}_{\overline{K}}$ and fix $i > 0$ and $j \geq 0$. We prove

$$H^i(\widetilde{X}_{\overline{K}}, \Omega^j_{\widetilde{X}_{\overline{K}}}(\overline{A})) = 0.$$

Let $A := \text{sp}(\overline{A})$. Then $\overline{A} \sim \widetilde{A}_{\overline{K}}$ for a lift $\widetilde{A}$ of A by the argument in (1). Since $\overline{A}$ is ample, we can take $m \gg 0$ such that $h^0(\widetilde{X}_{\overline{K}}, \mathcal{O}_{\widetilde{X}_{\overline{K}}}(m\widetilde{A}_{\overline{K}})) > 1$, and upper semi-continuity ([Har77, Theorem III.12.8]) shows that $h^0(X, \mathcal{O}_X(mA)) > 1$. Since the Picard number of X is 1, it follows that A is ample. By assumption, we have

$$H^i(X, \Omega^j_X(A)) = 0,$$

and upper semi-continuity shows the desired vanishing in characteristic zero. Thus, (3) holds.   □

We use Proposition 3.6 to reduce the following Proposition to the case of characteristic zero.

Proposition 3.8. *Let X be a smooth Fano threefold of Picard rank 1 over an algebraically closed field k such that X satisfies Bott vanishing. Then X is isomorphic to projective space.*

Proof. **Step 1** (characteristic zero case). Assume that k has characteristic zero. By Bott vanishing, we have $H^i(X, TX) = H^i(X, \Omega^2_X(-K_X)) = 0$ for all $i > 0$. By the Hirzebruch–Riemann–Roch theorem (cf. [AWZ23, Proof of Theorem 7.4]), we have

$$0 \leq h^0(X, TX) = \chi(X, TX) = \frac{1}{2}(-K_X)^3 - 18 + \rho(X) - h^1(X, \Omega^2_X),$$

where $\rho(X)$ is the Picard number of X. Then by [IP99, Tables 12.2], it follows that X is isomorphic to the quintic del Pezzo threefold V_5 (a smooth codimension-3 linear section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$), the quadric threefold Q, or $\mathbb{P}^3$. Since V_5 and Q do not satisfy Bott vanishing by [AWZ23, Lemma 7.10] and [BTLM97, subsection 4.1] (or Proposition 3.2), it follows that $X \cong \mathbb{P}^3$.

Step 2 (positive characteristic case). Assume that k has positive characteristic. Since X is Fano and satisfies Bott vanishing, we have

$$H^i(X, TX) = H^i(X, \mathcal{O}_X) = 0$$

for all $i > 0$. By [FGI$^+$05, Theorem 9.5.11], we can take a lift $\widetilde{X}$ over $W(k)$ of X, and the geometric generic fiber $\widetilde{X}_{\overline{K}}$ is a smooth Fano threefold. By Proposition 3.6 (1) and (3), the Picard number of $\widetilde{X}_{\overline{K}}$ is 1 and $\widetilde{X}_{\overline{K}}$ satisfies Bott vanishing. Thus, by Step 1, we have $\widetilde{X}_{\overline{K}} \cong \mathbb{P}^3_{\overline{K}}$. By Proposition 3.6(2), we have $r(X) = r(\widetilde{X}_{\overline{K}}) = 4$, and we conclude that $X \cong \mathbb{P}^3_k$ by Lemma 3.5. $\qquad\square$

3.3. **Images of toric varieties.** Let $f \colon Y \to X$ be a morphism from a projective toric variety Y onto a smooth projective variety X of Picard number 1. In characteristic zero, generalizing Lazarsfeld's result on images of projective space [Laz84, Theorem 4.1], Occhetta-Wiśniewski [OW02, Theorem 1.1] proved that X is isomorphic to projective space.

This result does not extend to characteristic $p > 0$ in full generality. For example, in characteristic 2, there is a finite purely inseparable morphism from $\mathbb{P}^3$ onto a smooth quadric threefold [Eke87, Proposition 2.5]. However, Occhetta-Wiśniewski's proof does work without change for *separable* morphisms in positive characteristic. That is:

Theorem 3.9. *Let X be a smooth projective variety of Picard number 1 over an algebraically closed field. Let Y be a proper toric variety. If there is a separable morphism $Y \to X$, then X is isomorphic to projective space.*

For possible future use, let us show how a special case of Theorem 3.9 follows from our arguments with Bott vanishing. One can also prove a version of Proposition 3.10 using Achinger–Witaszek–Zdanowicz's results on images of F-liftable varieties [AWZ21, Theorem 4.4.1]. For example, they showed that a smooth complex surface that is an image of a proper toric variety must be toric [AWZ21, Theorems 2 and 3].

Proposition 3.10. *Let X be a smooth projective threefold of Picard number 1 over an algebraically closed field k. Let Y be a proper toric variety of the same dimension. If there is a morphism $Y \to X$ of degree invertible in k, then X is isomorphic to projective space.*

Proof. Let $Y \to Y' \to X$ be the Stein factorization. Replacing Y by Y', we may assume that f is finite and Y is a normal toric variety, by [Tan22, Proposition 2.7].

Let A be an ample Cartier divisor on X, and fix $i > 0$ and $j \geq 0$. By the same proof as in Proposition 2.7 (pushing forward differential forms), we have a split injection

$$H^i(X, \Omega_X^j(A)) \hookrightarrow H^i(Y, \Omega_Y^{[j]}(f^*A)),$$

where $\Omega_Y^{[j]}$ is the sheaf of reflexive differentials, $(\Omega_Y^j)^{**}$. Since f^*A is ample, Bott vanishing for toric varieties gives that the group on the right is zero, and so the group on the left is zero. That is, X satisfies Bott vanishing.

Since f is separable, we have $f^*(-K_X) = -K_Y + R$ for an effective divisor R, the *ramification divisor* [Kol13, equation 2.41.2]. Since $-K_Y$ is big and the Picard number of X is 1, it follows that $-K_X$ is ample. Therefore, $X \cong \mathbb{P}^3$ by Proposition 3.8. □

3.4. **Fano fourfolds of index greater than 1.** In this subsection, we prove that projective space is the only Fano fourfold of Picard number 1 and Fano index greater than 1 that satisfies Bott vanishing.

Lemma 3.11. *Let $Y \subset \mathbb{P}^N$ be a smooth projective variety of dimension at least 2 over an algebraically closed field k of characteristic zero, and let X be a smooth hyperplane section. Assume that $-K_Y = \mathcal{O}_Y(b)$ with $b \geq 2$. Then the following hold.*

(1) *We have*
$$\chi(X, TX) = \chi(Y, TY) - \chi(Y, TY(-1)) - h^0(Y, \mathcal{O}(1)) + 1.$$

(2) *We have*
$$\chi(X, TX(-1)) = \chi(Y, TY(-1)) - \chi(Y, TY(-2)) - 1.$$

(3) *For $2 \leq a \leq b-1$, we have*
$$\chi(X, TX(-a)) = \chi(Y, TY(-a)) - \chi(Y, TY(-a-1)).$$

Proof. We have exact sequences of coherent sheaves:
$$0 \to TY(-1) \to TY \to TY|_X \to 0 \text{ and}$$
$$0 \to TX \to TY|_X \to \mathcal{O}_X(1) \to 0.$$

For any integer a, it follows that
$$\chi(X, TX(-a)) = \chi(Y, TY(-a)) - \chi(Y, TY(-a-1)) - \chi(X, \mathcal{O}_X(1-a)).$$

Since $K_Y = \mathcal{O}_Y(-b)$, adjunction gives that $K_X = (K_Y + X)|_X = \mathcal{O}_X(1-b)$, and so X is Fano. By Kodaira vanishing, we have $H^{>0}(X, \mathcal{O}_X(1-a)) = 0$ for all $a \leq b-1$. Also, since $\mathcal{O}_X(1)$ is ample, we have $H^0(X, \mathcal{O}_X(1-a)) = 0$ for all $a \geq 2$, whereas $h^0(X, \mathcal{O}_X(1-a)) = 1$ for $a = 1$. Therefore, $\chi(X, \mathcal{O}_X(1-a))$ is zero for $2 \leq a \leq b-1$, and it is 1 for $a = 1$, proving statements (2) and (3). For $a = 0$, the exact sequence $0 \to \mathcal{O}_Y \to \mathcal{O}_Y(1) \to \mathcal{O}_X(1) \to 0$ shows that $\chi(X, \mathcal{O}_X(1)) = \chi(Y, \mathcal{O}_Y(1)) - 1 = h^0(Y, \mathcal{O}_Y(1)) - 1$, proving (1). □

Proposition 3.12. *Let X be a smooth Fano fourfold of Picard number 1 and Fano index greater than 1 over an algebraically closed field k. If X satisfies Bott vanishing, then X is isomorphic to projective space.*

Fano index	$\mathcal{O}(1)^4$	$\chi(X, TX)$	Description
2	2	-185	sextic in $\mathbb{P}^5(1^5 3)$
2	4	-90	quartic in $\mathbb{P}^5$
2	6	-55	complete intersection of quadric and cubic in $\mathbb{P}^6$
2	8	-36	complete intersection of three quadrics in $\mathbb{P}^7$
2	10	-24	$\mathrm{Gr}(2,5) \cap Q \cap \mathbb{P}^7 \subset \mathbb{P}^9$
2	12	-15	$\mathrm{OGr}(5,10)^+ \cap \mathbb{P}^9 \subset \mathbb{P}^{15}$
2	14	-9	$\mathrm{Gr}(2,6) \cap \mathbb{P}^{10} \subset \mathbb{P}^{14}$
2	16	-3	$\mathrm{LGr}(3,6) \cap \mathbb{P}^{11} \subset \mathbb{P}^{13}$
2	18	1	$G_2/P \cap \mathbb{P}^{12} \subset \mathbb{P}^{13}$
3	1	-103	sextic in $\mathbb{P}^5(1^4 23)$
3	2	-45	quartic in $\mathbb{P}^5(1^5 2)$
3	3	-20	cubic in $\mathbb{P}^5$
3	4	-4	complete intersection of two quadrics in $\mathbb{P}^6$
3	5	8	$\mathrm{Gr}(2,5) \cap \mathbb{P}^7 \subset \mathbb{P}^9$
4	2	15	quadric $Q \subset \mathbb{P}^5$
5	1	24	$\mathbb{P}^4$

TABLE 1. The Fano 4-folds of Picard number 1 with Fano index ≥ 2

Proof. If k has characteristic $p > 0$, then (as in the proof of Proposition 3.8) X lifts to characteristic 0, and the lift also satisfies Bott vanishing. So it suffices to prove the proposition for k of characteristic zero. (That will imply that the lift has Fano index 5, so X in characteristic p has Fano index 5 and satisfies Kodaira vanishing; so it is isomorphic to $\mathbb{P}^4_k$.)

So assume that k has characteristic zero. Then the smooth Fano fourfolds of Picard number 1 and index greater than 1 were classified by Fujita, Mukai, and Wilson [KP23, Theorem 1.2], [Muk89, Theorem 2], [Wil87]. The classification is listed in Table 1, where the calculations of $\chi(X, TX)$ can be made using Lemma 3.11. Since we assume that X satisfies Bott vanishing, the tangent bundle $TX = \Omega_X^3(-K_X)$ has zero cohomology in positive degrees, and so $\chi(X, TX) \geq 0$. (Here we used that on a smooth d-dimensional variety X, we have a dual pairing $\Omega_X^1 \times \Omega_X^{d-1} \to \Omega_X^d = \mathcal{O}(K_X)$, and so $TX \cong \Omega^{d-1}(-K_X)$.) By Table 1, X is either $\mathbb{P}^4$, the quadric 4-fold Q, a codimension-2 linear section of the Grassmannian $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$, or a hyperplane section of the G_2-Grassmannian $G_2/P \subset \mathbb{P}^{13}$. In each of these cases other than $\mathbb{P}^4$, we will disprove Bott vanishing by a Riemann-Roch calculation.

We know that Bott vanishing fails for the quadric fourfold, as mentioned in Proposition 3.2. It remains to disprove Bott vanishing for the other two fourfolds above. First, let Y be $G_2/P \subset \mathbb{P}^{13}$; then Y has dimension 5 and $-K_Y = \mathcal{O}_Y(3)$. Using the Borel-Weil-Bott theorem, Konno showed (in characteristic zero, as here) that $TY(-a)$ has zero cohomology in all degrees for $1 \leq a \leq 2$ [Kon89, Theorem 3.4.1]. By Lemma 3.11, the Fano fourfold $X = G_2/P \cap \mathbb{P}^{12} \subset \mathbb{P}^{13}$ has $\chi(X, TX(-1)) = 0 - 1 = -1 < 0$.

By adjunction, we have $-K_X = \mathcal{O}(2)$, so $\chi(X, \Omega^3(1)) = \chi(X, TX(-1)) = -1 < 0$, and so X does not satisfy Bott vanishing.

Finally, let Y be $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$, which has dimension 6 and Fano index 5. By applying Lemma 3.11 twice, a codimension-2 linear section X has

$$\chi(X, TX(-1)) = \chi(Y, TY(-1)) - 2\chi(Y, TY(-2)) + \chi(Y, TY(-3)) - 2.$$

Snow showed (in characteristic zero, as here) that if Z is a Grassmannian $\mathrm{Gr}(s,t)$ other than projective space or $\mathrm{Gr}(2,4)$, then $-K_Z = \mathcal{O}_Z(t)$ and $TZ(-a)$ has zero cohomology in all degrees for $1 \le a \le t-1$ [Sno86, Theorem 3.4(3)]. In particular, $Y = \mathrm{Gr}(2,5)$ has $\chi(Y, TY(-a)) = 0$ for $1 \le a \le 4$. By the formula above, the Fano fourfold $X = \mathrm{Gr}(2,5) \cap \mathbb{P}^7 \subset \mathbb{P}^9$ has $\chi(X, TX(-1)) = -2 < 0$. By adjunction, $-K_X = \mathcal{O}_X(3)$, So $\chi(X, \Omega^3(2)) = \chi(X, TX(-1)) = -2 < 0$, and so X does not satisfy Bott vanishing. This completes the proof that if a smooth Fano fourfold of Picard number 1 and Fano index greater than 1 satisfies Bott vanishing, then it is isomorphic to projective space. $\qquad\square$

4. Proof of Theorem A and Theorem B

Proof of Theorems A and B. In the situation of Theorem A (resp. B), X satisfies Bott vanishing by Theorem C (resp. Proposition 2.7), and the assertion follows from Propositions 3.2, 3.8, and 3.12. $\qquad\square$

5. Failure of Bott vanishing for separable polarized endomorphisms

We now show that Bott vanishing can fail if we only assume that an int-amplified endomorphism is separable (rather than of degree invertible in k). This resolves a question in the first version of this paper. (Bott vanishing obviously fails for inseparable endomorphisms, since *every* projective variety over a finite field has the Frobenius endomorphism, which is int-amplified.)

Proposition 5.1. *For any prime power q at least 4, there is a smooth projective 3-fold X over $\mathbb{F}_q$ such that X has a separable polarized (hence int-amplified) endomorphism, but Bott vanishing fails on X.*

Proof. Let q be a prime power at least 4, and let X be the blow-up of $\mathbb{P}^3$ over $\mathbb{F}_q$ at some set of $\mathbb{F}_q$-points on the plane $\{w = 0\}$. Then X has a separable polarized endomorphism (inspired by a 2-dimensional example by Nakayama [Nak10, Example 4.5]). Namely, consider the endomorphism of $\mathbb{P}^3$ given by

$$g([x, y, z, w]) = [x^q - xw^{q-1}, y^q - yw^{q-1}, z^q - zw^{q-1}, w^q].$$

Then g is separable, but it restricts to the Frobenius morphism on the plane $\{w = 0\}$. In particular, g fixes the given set of $\mathbb{F}_q$-points in the plane $\{w = 0\}$. A direct calculation shows that g lifts to an endomorphism f of the blow-up X. (It suffices to check this over the point $[1, 0, 0, 0]$ in $\mathbb{P}^3$, in view of the symmetry group $GL(3, \mathbb{F}_q)$ (acting on x, y, z) of the endomorphism g.) Clearly f is separable, since g is.

We now specialize to the case where X is the blow-up of $\mathbb{P}^3$ over $\mathbb{F}_q$ at 5 $\mathbb{F}_q$-points in the plane $\{w = 0\}$ with no 3 collinear. (This is possible for $q \ge 4$, as we

assumed.) These points are contained in a unique smooth conic C. After a change of coordinates in $GL(3, \mathbb{F}_q)$, we can assume that C is the conic $\{w = 0, xy = z^2\}$. Write H for the pullback to X of the line bundle $\mathcal{O}_{\mathbb{P}^3}(1)$ on $\mathbb{P}^3$, and $E_1, \ldots, E_5$ for the exceptional divisors. Then $f^*H = qH$ and $f^*E_j = qE_j$ for $j = 1, \ldots, 5$. Since $\mathrm{Pic}(X) = \mathbb{Z}\{H, E_1, \ldots, E_5\}$, f is polarized, hence int-amplified.

Next, we show that the line bundle $A := 3H - \sum_{j=1}^5 E_j$ is ample on X. Indeed, xw, yw, zw, w^2, and $xy - z^2$ are sections of $L := 2H - \sum E_j$, and so the base locus of L is only the strict transform of the conic C in $\mathbb{P}^3$. Also, L has positive degree on every curve in $E_1, \ldots, E_5$. The base locus of $A = L + H$ is at most the conic C, but the linear system $|A|$ also contains the sum of a plane through p_1 and p_2, a plane through p_1 and p_3, and a plane through p_4 and p_5; so A is basepoint-free on X. Since $A = L + H$ has $A \cdot C = 1$, A has positive degree on every curve on X. Together with basepoint-freeness, this implies that A is ample.

To disprove Bott vanishing on X, we will show that $H^1(X, \Omega^1_X(A))$ is not zero. Consider the exact sequence $0 \to \mathcal{O}_X(-\sum_j E_j) \to \mathcal{O}_X \to \oplus_j \mathcal{O}_{E_j} \to 0$. Tensoring with $\Omega^1_X(3H)$ and taking cohomology gives an exact sequence:

$$H^0(X, \Omega^1_X(3H)) \to \oplus_{j=1}^5 H^0(E_j, \Omega^1_X(3H)) \to H^1(X, \Omega^1_X(A)).$$

So it suffices to show that the restriction map on H^0 is not surjective. Since the line bundle $3H$ is pulled back from $\mathbb{P}^3$, the first space is isomorphic to $H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(3))$. Next, each E_j is isomorphic to $\mathbb{P}^2$, and we have an exact sequence on $E := E_j$: $0 \to \mathcal{O}_X(-E)|_E \to \Omega^1_X|_E \to \Omega^1_E \to 0$, where $\mathcal{O}_X(-E)|_E = \mathcal{O}_E(1)$ on $E = \mathbb{P}^2$. Here $H^0(E, \Omega^1_E) = 0$, and so $h^0(E, \Omega^1_X|_E) = h^0(E, \mathcal{O}_E(1)) = 3$. The line bundle H on X is trivial on each E_j, and so $h^0(E_j, \Omega^1_X(3H)) = 3$ for $j = 1, \ldots, 5$. More canonically, $H^0(E_j, \Omega^1_X(3H)) \cong H^0(p_j, \Omega^1_{\mathbb{P}^3}(3H))$. So, to disprove Bott vanishing on X, it suffices to show that the restriction map $H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(3H)) \to \oplus_{j=1}^5 H^0(p_j, \Omega^1_{\mathbb{P}^3}(3H))$ has rank less than $5 \cdot 3 = 15$.

The point is that this restriction map factors through $H^0(C, \Omega^1_{\mathbb{P}^3}(3H))$, since $p_1, \ldots, p_5$ lie on the conic C. To analyze that group, note that the vector bundle $\Omega^1_{\mathbb{P}^n}(2H)$ is globally generated for any n [Laz04, equation 7.13]. As a result, $\Omega^1_{\mathbb{P}^3}(3H)$ is ample, so its restriction to $C \cong \mathbb{P}^1$ is ample, and hence $H^1(C, \Omega^1_{\mathbb{P}^3}(3H)) = 0$. By Riemann-Roch, it follows that

$$\begin{aligned}
h^0(C, \Omega^1_{\mathbb{P}^3}(3H)) &= \chi(C, \Omega^1_{\mathbb{P}^3}(3H)) \\
&= \deg_C(\Omega^1_{\mathbb{P}^3}(3H)) + \mathrm{rank}(\Omega^1_{\mathbb{P}^3}(3H))(1 - g(C)) \\
&= 10 + 3 = 13.
\end{aligned}$$

So the restriction map $H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(3H)) \to \oplus_{j=1}^5 H^0(p_j, \Omega^1_{\mathbb{P}^3}(3H))$ has rank at most 13, thus less than 15. By the previous paragraph, this completes the proof that Bott vanishing fails for X, even though X has a separable polarized endomorphism. $\square$

6. Global F-regularity of Fano varieties with an endomorphism

Let X be a Fano variety in characteristic $p > 0$ that is strongly F-regular (for example, smooth). If X admits an int-amplified endomorphism of degree prime to p,

we will show that X is globally F-regular (Theorem D). (It was known to the experts that a smooth Fano variety satisfying Bott vanishing must be globally F-split, by the argument sketched in [BK05, Exercise 1.6.4]. Therefore, when X is smooth, Theorem D is an immediate consequence of Theorem C.) Intuitively, "strongly F-regular" is a strong version of "klt type" in characteristic p, and "globally F-regular" is a strong version of "Fano type".

Definition 6.1. Let X be a normal variety over a perfect field k of characteristic $p > 0$, and let B be an effective $\mathbb{Q}$-divisor on X.

(1) The pair (X, B) is *globally F-regular* if for every effective Weil divisor D on X, there is a positive integer e such that the composite map

$$\mathcal{O}_X \to F_*^e \mathcal{O}_X \hookrightarrow F_*^e \mathcal{O}_X(\lceil (p^e - 1)B \rceil + D)$$

splits as an $\mathcal{O}_X$-module homomorphism [SS10, Definition 3.1]. (Note that $F_*^e \mathcal{O}_X(Z)$ means $F_*^e(\mathcal{O}_X(Z))$, for a divisor Z.)

(2) The pair (X, B) is *globally sharply F-split* if there is a positive integer e such that the composite map

$$\mathcal{O}_X \to F_*^e \mathcal{O}_X \hookrightarrow F_*^e \mathcal{O}_X(\lceil (p^e - 1)B \rceil)$$

splits as an $\mathcal{O}_X$-module homomorphism. (For $B = 0$, we omit the word "sharply".)

(3) The pair (X, B) is *strongly F-regular*, resp. *sharply F-pure*, if X is covered by open sets on which the corresponding global property holds. (For $B = 0$, we simply say F-*pure* to mean "sharply F-pure.")

To avoid confusion, note that whether the map $\mathcal{O}_X \to F_*^e \mathcal{O}_X(Z)$ splits depends on the effective divisor Z, not just on its linear equivalence class.

Let X be a smooth variety over a perfect field k of characteristic $p > 0$. The Frobenius pushforward of the de Rham complex,

$$F_* \Omega_X^\bullet : F_* \mathcal{O}_X \overset{F_* d}{\to} F_* \Omega_X \overset{F_* d}{\to} \cdots ,$$

is a complex of $\mathcal{O}_X$-module homomorphisms. Define locally free $\mathcal{O}_X$-modules as follows.

$$B_X^i := \mathrm{im}(F_* d : F_* \Omega_X^{i-1} \to F_* \Omega_X^i),$$
$$Z_X^i := \ker(F_* d : F_* \Omega_X^i \to F_* \Omega_X^{i+1}).$$

By definition, we have an exact sequence

(6.1.1) $$0 \to Z_X^i \to F_* \Omega_X^i \overset{F_* d}{\to} B_X^{i+1} \to 0.$$

We also have the exact sequence arising from the Cartier isomorphism (see [BK05, Theorem 1.3.4], for example),

(6.1.2) $$0 \to B_X^i \to Z_X^i \overset{C^i}{\to} \Omega_X^i \to 0.$$

Theorem 6.2. *Let X be a Fano variety over a perfect field of characteristic $p > 0$. Suppose that X admits an int-amplified endomorphism of degree prime to p.*

(1) *If X is strongly F-regular (for example, smooth), then it is globally F-regular.*

(2) *If X is F-pure, then it is globally F-split.*

Remark 6.3. Theorem 6.2 is sharp in some ways. Consider the projective cone $X \subset \mathbb{P}^3$ over a smooth cubic curve $C \subset \mathbb{P}^2$ over an algebraically closed field k of characteristic $p > 0$. Then X is a log canonical Fano surface, and it admits an int-amplified endomorphism of degree prime to p, coming from a multiplication endomorphism of the elliptic curve C. But X is not strongly F-regular, hence not globally F-regular. And if C is supersingular, then X is not F-pure, hence not globally F-split. One might ask: is a klt Fano variety with an int-amplified endomorphism of degree prime to p always globally F-regular?

Proof. (Theorem 6.2) We first prove that global F-regularity of a Fano variety is equivalent to global F-splitting plus strong F-regularity, using the results of Schwede and Smith. As a result, statement (2) will imply statement (1).

Lemma 6.4. *Let X be a normal quasi-projective variety over a perfect field k, and let B be an effective $\mathbb{Q}$-divisor on X. If the pair (X, B) is globally sharply F-split, then there is an effective $\mathbb{Q}$-divisor Δ such that $(X, B+\Delta)$ is globally sharply F-split, $B + \Delta$ has $\mathbb{Z}_{(p)}$ coefficients, and $K_X + B + \Delta$ is $\mathbb{Z}_{(p)}$-linearly equivalent to zero.*

This is [SS10, Theorem 4.3]. They do not mention that $K_X + B + \Delta$ is $\mathbb{Z}_{(p)}$-linearly equivalent to zero, but that is what their proof gives (p. 878).

Let X be a strongly F-regular Fano variety that is globally F-split. Then Lemma 6.4 gives an effective $\mathbb{Z}_{(p)}$-divisor Δ such that (X, Δ) is globally F-split and $K_X + \Delta$ is $\mathbb{Z}_{(p)}$-linearly equivalent to zero. By the definition of global F-splitting, there is a positive integer e such that $(p^e - 1)\Delta$ has integer coefficients and the inclusion $\mathcal{O}_X \to F_*^e \mathcal{O}_X((p^e-1)\Delta)$ is split. Here Δ is $\mathbb{Z}_{(p)}$-linearly equivalent to $-K_X$, which is ample. So $X - \mathrm{Supp}(\Delta)$ is affine and strongly F-regular, hence globally F-regular. By [SS10, Theorem 3.9], it follows that X is globally F-regular.

It remains to prove statement (2). That is, if X is an F-pure Fano variety that admits an int-amplified endomorphism of degree prime to p, we will show that X is globally F-split. Since X is F-pure, the exact sequence

$$0 \to \mathcal{O}_X \to F_* \mathcal{O}_X \to F_* \mathcal{O}_X / \mathcal{O}_X \to 0$$

is locally split on X. In particular, the sheaf $F_* \mathcal{O}_X / \mathcal{O}_X$ is reflexive, since $F_* \mathcal{O}_X$ is. On the smooth locus U of X, we have $0 \to \mathcal{O}_U \to F_* \mathcal{O}_U \to B_U^1 \to 0$. So $F_* \mathcal{O}_X / \mathcal{O}_X$ is the double dual $B_X^{[1]}$ of B_X^1.

Since the sequence

$$0 \to \mathcal{O}_X \to F_* \mathcal{O}_X \to B_X^{[1]} \to 0$$

is locally split, it corresponds to an element of $H^1(X, \mathcal{H}om(B_X^{[1]}, \mathcal{O}_X))$. We want to show that X is globally F-split, meaning that this element is zero. We have a perfect pairing $B_U^1 \times B_U^d \to \omega_U$ on the smooth locus U of X [MS87, proof of Lemma 1.1]. As a result, the sheaf $\mathcal{H}om(B_X^{[1]}, \mathcal{O}_X)$ is the reflexive sheaf $B_X^{[d]}(-K_X)$. So it suffices to show that $H^1(X, B_X^{[d]}(-K_X)) = 0$. That follows from Lemma 6.5, below. So X is globally F-split, proving statement (2). Theorem 6.2 is proved. $\square$

Lemma 6.5. *Let X be a normal projective variety over a perfect field of characteristic $p > 0$. Suppose that X admits an int-amplified endomorphism of degree prime to p. Then*

$$H^i(X, B_X^{[j]}(A)) = 0$$

and

$$H^i(X, Z_X^{[j]}(A)) = 0$$

for every $i > 0$, $j \geq 0$, and A an ample Weil divisor.

Proof. The proof of Theorem C works without change for the reflexive sheaves $B_X^{[j]}$ and $Z_X^{[j]}$ in place of $\Omega_X^{[j]}$. In more detail, consider the pullback map $\Omega_X^{[j]} \to f_*\Omega_X^{[j]}$ and the trace map $\tau_f \colon f_*\Omega_X^{[j]} \to \Omega_X^{[j]}$. Because f commutes with the Frobenius morphism F on X, we also have a pullback map $F_*\Omega_X^{[j]} \to f_*F_*\Omega_X^{[j]}$ and a trace map $\tau_f \colon f_*F_*\Omega_X^{[j]} \to F_*\Omega_X^{[j]}$. We claim that these two maps preserve the subsheaves $B_X^{[j]}$ and $Z_X^{[j]}$ of $F_*\Omega_X^{[j]}$; then the proof of Theorem C applies.

Since $B_X^{[j]}$ and $Z_X^{[j]}$ are reflexive sheaves, it suffices to check this claim outside $X^{\mathrm{sing}} \cup f(X^{\mathrm{sing}})$. Then the claim follows from the fact that the pullback and pushforward of differential forms commute with the exterior derivative d [SPA23, Tag 0FLC].   □

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