The curvature of a Hessian metric

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Given a smooth function f on an open subset of a real vector space, one can define the associated "Hessian metric" using the second derivatives of f,

$$g_{ij} := \partial^2 f/\partial x_i \partial x_j.$$

In this paper, inspired by P.M.H. Wilson's paper on sectional curvatures of Kähler moduli [31], we concentrate on the case where f is a homogeneous polynomial (also called a "form") of degree d at least 2. Following Okonek and van de Ven [23], Wilson considers the "index cone," the open subset where the Hessian matrix of f is Lorentzian (that is, of signature $(1,*)$) and f is positive. He restricts the indefinite metric $-1/d(d-1)\partial^2 f/\partial x_i \partial x_j$ to the hypersurface $M := \{f = 1\}$ in the index cone, where it is a Riemannian metric, which he calls the Hodge metric. (In affine differential geometry, this metric is known as the "centroaffine metric" of the hypersurface M, up to a constant factor.) Wilson considers two main questions about the Riemannian manifold M. First, when does M have nonpositive sectional curvature? (It does have nonpositive sectional curvature in many examples.) Next, when does M have constant negative curvature?

On the first question, Wilson gave examples of cubic forms f on $\mathbf{R}^3$ to show that the surface M need not have nonpositive curvature everywhere. But he showed that for every cubic form on $\mathbf{R}^3$ such that M is nonempty (that is, the index cone is nonempty), M has nonpositive curvature somewhere ([31], Prop. 5.2). One result of this paper is to confirm Wilson's suggestion that this statement should fail for forms of higher degree or on a higher-dimensional space. Namely, we give examples of a quartic form on $\mathbf{R}^3$ and a cubic form on $\mathbf{R}^4$ such that M is nonempty and M has positive sectional curvature on some 2-plane at every point (Lemmas 4.1 and 5.1). If Wilson's conjecture that the Kähler moduli space of a Kähler manifold has nonpositive sectional curvature is correct, then these forms cannot occur as the intersection form on $H^{1,1}(X, \mathbf{R})$ for a Kähler 4-fold with $h^{1,1} = 3$, or a Kähler 3-fold with $h^{1,1} = 4$ (respectively), although they would be allowed by the Hodge index theorem.

Wilson showed that the Riemannian manifold M has constant negative sectional curvature when f is a Fermat form $x_1^d - x_2^d - \cdots - x_n^d$ ([31], Introduction, Example 2). More generally, we show that M has constant negative curvature $-d^2/4$ when f is a sum of forms of degree d in at most two variables, $f = \alpha_1(x_1, x_2) + \alpha_2(x_3, x_4) + \cdots$. The problem of finding forms f such that the surface M has constant curvature is a special case of the WDVV equations of string theory, as explained in section 2. In fact, section 2 lists a whole series of natural problems of differential geometry that are essentially equivalent to the WDVV equations.

The problem of finding all forms f on $\mathbf{R}^3$ such that the surface M has constant curvature $-d^2/4$ also has a close relation to classical invariant theory, in particular to the "Clebsch covariant" $S(f)$ studied by Clebsch [3] and Dolgachev-Kanev [4]; see Question 6.3. Using these ideas, we prove that any ternary form f of degree at most 4 such that the surface M has constant curvature $-d^2/4$ is in the closure of the set of forms which can be written $f = \alpha(x, y) + \beta(z)$ in some linear coordinate system (Theorem 8.1); this was known from Wilson's results when f has degree 3 ([31], Examples, section 5). Also, we prove a weaker result for plane curves of any degree: the ternary forms which can be written $f = \alpha(x, y) + \beta(z)$ in some linear coordinate system always form an irreducible component of the set of all forms such that the associated surface M has constant curvature (Theorem 9.1).

Dubrovin showed that Maschke's ternary sextic $f = x^6 + y^6 + z^6 - 10(x^3y^3 + y^3z^3 + z^3x^3)$, the invariant of lowest degree for a complex reflection group of order 648 on $\mathbf{C}^3$, gives a surface M with constant curvature $-d^2/4$ ([7], Corollary 5.9 and Example 3). Dubrovin used the equivalent statement that this form gives a flat Hessian metric; see Corollary 2.3 below. The Maschke sextic is not in the closure of the set of forms which can be written as $\alpha(x, y) + \beta(z)$ in some linear coordinates. We can ask if the Maschke sextic is the only ternary form of any degree which gives a surface M of constant curvature $-d^2/4$ while not being in the closure of the forms $\alpha(x, y) + \beta(z)$ (Question 6.2). It may seem implausible that the Maschke sextic should be the sole exception here; in some sense, that would mean that complex reflection groups play a very special role in this problem. A rough analogy which encourages this belief is Hertling's theorem: any massive Frobenius manifold whose Euler vector field has positive degrees arises from some Coxeter group ([15], Theorem 5.25).

The paper starts with general formulas for the curvature of Hessian metrics. We relate Wilson's constructions to the literature on Hessian metrics, using the notion of warped products (Lemma 2.1).

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1 Conventions

Here is Wilson's definition of the "Hodge metric." Let X be a compact Kähler manifold of dimension d at least 2. The cup product on $H^2(X, \mathbf{Z})$ determines a degree d form $f(\omega) := \omega^d \in \mathbf{R}$ on $H^{1,1}(X, \mathbf{R})$, and the *positive cone* is defined to be the set of elements ω of $H^{1,1}(X, \mathbf{R})$ such that ω^d is positive. The cup product also determines an *index cone*, as defined by Okonek and van de Ven [23]: the set of elements ω in the positive cone such that the quadratic form on $H^{1,1}(X, \mathbf{R})$ defined by $L \mapsto \omega^{d-2}L^2$ has signature $(1, h^{1,1} - 1)$. The Hodge index theorem says that any Kähler class in $H^{1,1}(X, \mathbf{R})$ lies in the index cone. Let W_1 be the set of points ω in the index cone with $\omega^d = 1$. Then W_1 is a smooth manifold, whose tangent space at a point ω is the set of L in $H^{1,1}(X, \mathbf{R})$ such that $\omega^{d-1}L = 0$. The *Hodge metric* is the Riemannian metric on W_1 defined by, for tangent vectors L_1 and L_2 at a point ω in W_1,

$$(L_1, L_2) = -\omega^{d-2}L_1L_2.$$

One computes easily that this metric is the restriction to W_1 of the pseudo-Riemannian Hessian metric $-1/d(d-1)\partial^2 f/\partial x_i \partial x_j$ on $H^{1,1}(X, \mathbf{R})$. In this paper, we use $-1/d(d-1)\partial^2 f/\partial x_i \partial x_j$ as our basic pseudo-Riemannian metric on an open subset of a real vector space. We also study its restriction to the hypersurface $M := W_1 = \{f = 1\}$. We usually call this metric on $\mathbf{R}^n$ the Hessian metric associated to f. Outside this paper, that name is usually restricted to the metric $\partial^2 f/\partial x_i \partial x_j$ with no constant factor.

We sometimes use the notation f_i for $\partial f/\partial x_i$, f_{ij} for $\partial^2 f/\partial x_i \partial x_j$, and so on. Starting from section 6, we consider ternary forms $f(x, y, z)$, and we identify these variables with x_1, x_2, x_3, so that f_{23} denotes $\partial^2 f/\partial y \partial z$.

2 The Hessian metric of a homogeneous polynomial

We begin by recalling the formula for the curvature of the Hessian metric $g_{ij} := \partial^2 f/\partial x_i \partial x_j$ associated to a smooth function f on a domain in $\mathbf{R}^n$. When f is a homogeneous polynomial, the metric on a domain in $\mathbf{R}^n$ is determined in a simple way, as a warped product, from its restriction to the hypersurface $M := \{f = 1\}$. We deduce a formula for the curvature of M, using the known result on $\mathbf{R}^n$; this approach is simpler than trying to compute the curvature directly on M. Finally, we recall the known relation of the metric on M with another Hessian metric on $\mathbf{R}^n$, $\partial^2(\log f)/\partial x_i \partial x_j$, and with the "centroaffine metric" of M. Along the way, we relate these constructions to the WDVV equations, although this is not needed for the rest of the paper.

Hessian metrics have also been called affine Kähler metrics ([2], [21], [18]), since any Kähler metric on a complex manifold has an analogous local description as $\partial^2 f/\partial z_i \partial \overline{z_j}$. But the name "Hessian" seems preferable on historical grounds.

Hessian metrics are a very natural way to construct Riemannian or pseudo-Riemannian metrics. For example, they have been used to define a canonical Riemannian metric on an arbitrary convex cone [30]. A variant of Hessian metrics can be used to define a canonical Riemannian metric on any convex domain, using the solution of a Monge-Ampère equation by Loewner-Nirenberg and Cheng-Yau ([20], [1], [26]). There are recent surveys on Hessian metrics by Duistermaat [9] and Shima-Yagi [27]. Duistermaat observed that one canonical Hessian metric on a convex domain can have positive curvature ([8], 8.4); that phenomenon is roughly analogous to some examples below (Lemmas 4.1 and 5.1).

There is a geometric interpretation of Hessian metrics which has appeared recently in mirror symmetry (Hitchin [16], Leung [19], Kontsevich-Soibelman [18]). It is worth mentioning, although it will only be used in this section of the paper. Given a real vector space V, the cotangent bundle $T^*V = V \oplus V^*$ has a natural symplectic form $\sum_i dx_i \wedge dp_i$, as is well known, but also a natural pseudo-Riemannian metric, $\sum_i dx_i dp_i$. Consider any Lagrangian submanifold of T^*V. Such a submanifold, if its tangent space is in general position, can be viewed locally as the graph of an exact 1-form df, for a smooth function f on an open subset of V. Then the restriction of the pseudo-Riemannian metric on T^*V to this submanifold is precisely the Hessian metric associated to f. Generically, a Lagrangian submanifold of T^*V can also be viewed as the graph of $d\hat{f}$ for a function $\hat{f}$ on an open subset of V^*, known as the Legendre transform of f. So this picture "explains" the classical fact that a function and its Legendre transform determine isometric Hessian metrics.

To begin our computations, let f be a smooth function on a region in $\mathbf{R}^n$, and define $g_{ij} = \partial^2 f/\partial x_i \partial x_j$. We assume that $\det(g_{ij})$ is not zero; then g_{ij} is a pseudo-Riemannian metric. (O'Neill's book [25] is a convenient reference for pseudo-Riemannian metrics.) The formula for the curvature tensor of a Hessian metric is well known. The easiest way to compute it uses the following classical formulas for any pseudo-Riemannian metric, giving the Christoffel symbols of the first kind and the curvature tensor ([28], Chapter 4.D, equation (***)):

$$\Gamma_{ijk} = \frac{1}{2}(g_{ij,k} + g_{jk,i} - g_{ik,j})$$
$$R_{ijkl} = -\frac{1}{2}(g_{ik,jl} + g_{jl,ik} - g_{il,jk} - g_{jk,il}) - \sum_{p,q} g^{pq}(\Gamma_{jpl}\Gamma_{iqk} - \Gamma_{ipl}\Gamma_{jqk}),$$

where g^{ij} is the inverse of the matrix g_{ij}, $g_{ij,k}$ means $\partial g_{ij}/\partial x_k$, and so on. For a Hessian metric, it follows immediately that $\Gamma_{ijk} = f_{ijk}/2$ and

$$R_{ijkl} = -\frac{1}{4}\sum_{p,q} g^{pq}(f_{jlp}f_{ikq} - f_{ilp}f_{jkq}).$$

It is remarkable that the curvature of a Hessian metric depends only on the derivatives of f to order at most three, whereas one would expect fourth derivatives of f to come in; Duistermaat gives some explanation for this phenomenon [9].

With our conventions, the sectional curvature of the 2-plane spanned by $\partial/\partial x_1$ and $\partial/\partial x_2$ is $R_{1212}/(g_{11}g_{22} - g_{12}^2)$.

The second half of this paper (see Question 6.2) will be devoted to the study of flat Hessian metrics, that is, functions f such that the expression R_{ijkl} is identically zero. This system of partial differential equations looks much like the WDVV equations of string theory,

$$\sum_{p,q}(f_{jlp}f_{ikq} - f_{ilp}f_{jkq}) = 0.$$

In fact, Kito showed that the problem of finding flat Hessian metrics is precisely equivalent (by changing to coordinates adapted to the metric) to the WDVV equations ([17], Lemma 2.2). By the interpretation of Hessian metrics in terms of Lagrangian submanifolds discussed earlier, it follows that the WDVV equations also describe the flat Lagrangian submanifolds of $\mathbf{R}^{2n}$, where $\mathbf{R}^{2n}$ is given a natural symplectic structure and pseudo-Riemannian metric of signature (n, n).

In fact, there is a whole series of natural problems of differential geometry that are essentially equivalent to the WDVV equations. The problem of classifying flat Lagrangian submanifolds in $\mathbf{R}^{2n}$ with the above pseudo-Riemannian metric is equivalent to the WDVV equations, as we have just mentioned, but another real form of the same problem is the classification of flat Lagrangian submanifolds of the Kähler manifold $\mathbf{C}^n$. Terng showed that the latter problem is essentially equivalent to a natural integrable system of first-order PDEs which she defined, the $U(n)/O(n)$-system ([29], Prop. 3.5.3). Yet another equivalent problem is that of finding Egorov metrics, that is, flat metrics of the form: $g_{ij} = 0$ for $i \neq j$ and $g_{ii} = \partial f/\partial x_i$ ([29], Theorem 3.4.3). Finally, Ferapontov found that the problem in affine differential geometry of classifying hypersurfaces in $\mathbf{R}^n$ whose centroaffine metric is flat also reduces to the WDVV equations [11]. There is a similar reduction of hypersurfaces whose centroaffine metric has nonzero constant curvature to WDVV, as follows from Corollary 2.3 and the comments after it.

One basic result about this family of problems is Moore and Morvan's theorem that flat Lagrangian submanifolds of $\mathbf{C}^n$ are classified locally by $n(n+1)/2$ functions of one variable [22]. It follows, for example, that flat Hessian metrics on $\mathbf{R}^n$ are likewise classified locally by $n(n+1)/2$ functions of one variable.

The solutions of the WDVV equations which satisfy certain normalization and homogeneity conditions are described geometrically by Dubrovin's theory of Frobenius manifolds [5]. The most important Frobenius manifolds, the semisimple ones, are locally classified by finitely many numbers (rather than functions), although non-semisimple Frobenius manifolds of dimension at least 4 can depend on arbitrary functions of one variable ([6], Exercise 3.1). Dubrovin also discovered the relation of Frobenius manifolds with some of the integrable systems mentioned above, such as Egorov metrics.

Now suppose that the domain U in $\mathbf{R}^n$ is a cone (that is, U is preserved under multiplication by positive real numbers). Suppose also that f is homogeneous of some degree $d > 1$, in the sense that

$$f(\lambda x) = \lambda^d f(x)$$

for all $\lambda > 0$. Finally, assume that $f > 0$ on U.

From here on in the paper, we will use the metric $g_{ij} = -1/d(d-1)\partial^2 f/\partial x_i \partial x_j$ on U, to conform to Wilson's conventions as explained in section 1. The curvature tensor of that metric follows from the formula above:

$$R_{ijkl} = -\frac{1}{4d^2(d-1)^2} \sum_{p,q} g^{pq}(f_{jlp}f_{ikq} - f_{ilp}f_{jkq}),$$

where g^{ij} now denotes the inverse of the new matrix g_{ij}.

Let M be the hypersurface $\{f = 1\}$. Our assumption that f is homogeneous of degree > 1 implies that the restriction of the Hessian metric to M is nondegenerate, by the calculations in the following lemma. This lemma shows that the metric on M determines the metric on the whole open set U in a simple way. For any pseudo-Riemannian manifold M, define the *warped product* $(-1)\mathbf{R}^{>0} \times_t M$ to be the product manifold $\mathbf{R}^{>0} \times M$ with metric such that $T(\mathbf{R}^{>0})$ is orthogonal to TM everywhere, the inner product on TM at $(t, x) \in \mathbf{R}^{>0} \times M$ is t^2 times the given inner product, and the metric on $(-1)\mathbf{R}^{>0}$ is the negative definite metric $\langle \partial/\partial t, \partial/\partial t \rangle = -1$. (This construction is most familiar when M is the sphere S^{n-1} of radius 1: then the warped product $\mathbf{R}^{>0} \times_t S^{n-1}$ is isometric to flat Euclidean space $\mathbf{R}^n - 0$.) Also, for any pseudo-Riemannian manifold M, let aM denote the same manifold with inner product multiplied by a, $\langle x, y \rangle_{aM} = a\langle x, y \rangle_M$.

Lemma 2.1 *Give U the pseudo-Riemannian metric $-1/d(d-1)\partial^2 f/\partial x_i \partial x_j$, and give the hypersurface $M = \{f = 1\}$ the restricted metric. Then the map*

$$\varphi(t, x) := t^{2/d} x$$

gives an isometry from the warped product $(4/d^2)((-1)\mathbf{R}^{>0} \times_t (d^2/4)M)$ to U.

Proof. We use the Euler identity repeatedly: for a homogeneous function f of degree d, we have

$$\sum x_i f_i = df,$$

where f_i denote the derivatives of f at a point $x = (x_1, \ldots, x_n)$. Clearly φ gives a diffeomorphism from $\mathbf{R}^{>0} \times M$ to U.

Let v be a tangent vector to the hypersurface M in U at a point x. Clearly the metric $-1/d(d-1)\partial^2 f/\partial x_i \partial x_j$ on U is homogeneous of degree $d-2$. That is,

$$\langle v, v\rangle_{\lambda x} = \lambda^{d-2}\langle v, v\rangle_x$$

for any $\lambda > 0$, where the subscripts denote the point in U where the inner product is taken. Therefore,

$$\langle \lambda v, \lambda v\rangle_{\lambda x} = \lambda^{d}\langle v, v\rangle_x.$$

Pulling the metric on U back to $\mathbf{R}^{>0} \times M$ by the map $\varphi(t, x) = t^{2/d}x$ gives a metric such that

$$\langle v, v\rangle_{(t,x)} = t^2\langle v, v\rangle_{(1,x)}.$$

The warped product metric on $(4/d^2)((-1)\mathbf{R}^{>0} \times_t (d^2/4)M)$ has the same homogeneity property, and agrees with the given metric on M when $t = 1$. So the two metrics are the same on tangent vectors in TM at any point (t, x) in $\mathbf{R}^{>0} \times M$.

To check that $T\mathbf{R}$ is orthogonal to TM in the metric pulled back from U, we compute the inner product at a point $x \in U$ of the outward vector $\sum x_i \partial/\partial x_i$ with a tangent vector $\sum v_i \partial/\partial x_i$ to the hypersurface $f = c$, which means that $\sum v_i f_i = 0$:

$$\begin{aligned}
\langle \sum x_i \partial/\partial x_i, \sum v_i \partial/\partial x_i\rangle &= -1/d(d-1)\sum_{i,j} x_i v_j f_{ij} \\
&= -1/d \sum_j v_j f_j \\
&= 0.
\end{aligned}$$

Finally, we compute the inner product of the outward vector $\sum x_i \partial/\partial x_i$ at a point $x \in U$ with itself:

$$\begin{aligned}
\langle \sum x_i \partial/\partial x_i, \sum x_i \partial/\partial x_i\rangle &= -1/d(d-1)\sum_{i,j} x_i x_j f_{ij} \\
&= -1/d \sum_j x_j f_j \\
&= -f(x).
\end{aligned}$$

We compute that

$$\frac{\partial}{\partial t}\varphi(t, y) = \frac{2}{td}\varphi(t, y).$$

Using that, let us compute the length squared of the tangent vector $\partial/\partial t$ at the point (t, y) in $\mathbf{R}^{>0} \times M$ in the metric pulled back from U. Let $x = \varphi(t, y)$ in U.

$$\begin{aligned}
\langle \partial/\partial t, \partial/\partial t\rangle &= \frac{4}{d^2 t^2}\langle \sum x_i \partial/\partial x_i, \sum x_i \partial/\partial x_i\rangle \\
&= -\frac{4}{d^2 t^2} f(x) \\
&= -\frac{4}{d^2}.
\end{aligned}$$

This agrees with the inner product $\langle \partial/\partial t, \partial/\partial t\rangle$ in the warped product metric $(4/d^2)((-1)\mathbf{R}^{>0} \times_t (d^2/4)M)$, as we want. QED

Using O'Neill's formula for the sectional curvature of a warped product of pseudo-Riemannian manifolds, we now deduce a simple relation between the curvature of the open set U in $\mathbf{R}^n$ and the hypersurface M.

Corollary 2.2 *Let x be a point in the hypersurface M in U, P a nondegenerate 2-plane in the tangent space to M at x, and u a positive real number. Let $K_M(P)$ be the sectional curvature of M at the 2-plane P. Then the sectional curvature of U at the point cx and the 2-plane cP is*

$$K_U(cP) = \frac{1}{c^d}(K_M(P) + d^2/4).$$

Proof. This follows from Lemma 2.1 together with O'Neill's curvature formula for a warped product ([25], Proposition 7.42). QED

Corollary 2.3 *The pseudo-Riemannian Hessian metric $-1/d(d-1)\partial^2 f/\partial x_i \partial x_j$ on U is flat if and only if its restriction to the hypersurface $M := \{f = 1\}$ has constant sectional curvature $-d^2/4$.*

Proof. Suppose M has constant sectional curvature $-d^2/4$. Then $-(d^2/4)M$ has constant sectional curvature 1, and so the warped product $\mathbf{R}^{>0} \times_t (-d^2/4)M$ is flat by O'Neill's formulas ([25], Proposition 7.42). (The formulas are the same as those showing that the warped product $\mathbf{R}^{>0} \times_t S^{n-1}$ is isometric to $\mathbf{R}^n - 0$.) By Lemma 2.1, it follows that U is flat. The converse is immediate from Corollary 2.2. QED

Wilson already noticed that the value $-d^2/4$ for the curvature of M plays a special role, and Corollary 2.3 provides an explanation of this phenomenon.

To conclude this section, I will state the relation between the Hessian metric $\partial^2(\log f)/\partial x_i \partial x_j$ on U and the above metric on the hypersurface M, which is proved by the same kind of calculation as Lemma 2.1 (Loftin [21], Theorem 1). Loftin also mentions that the above metric on M (with a different normalization, namely the metric $-1/d\ \partial^2 f/\partial x_i \partial x_j$ restricted to M) is known in affine differential geometry as the *centroaffine metric* of M. This is not needed for the rest of the paper, but the Hessian metric associated to the logarithm of a homogeneous function is used in many papers on convex cones ([30], [13], [14]).

Lemma 2.4 *Give $M = \{f = 1\}$ the pseudo-Riemannian metric obtained by restricting the Hessian metric $-\partial^2 f/\partial x_i \partial x_j$ on U. Then the map*

$$\alpha(t, x) := e^{t/\sqrt{d}} x$$

is an isometry from the product $\mathbf{R} \times M$ to the Hessian metric $-\partial^2(\log f)\partial x_i \partial x_j$ on U. Here the real line $\mathbf{R}$ is given its usual Riemannian metric.

Example. The most famous examples of Hessian metrics are those associated to the "symmetric cones". Namely, let f be either a Lorentzian quadratic form $x_1^2 - x_2^2 - \cdots - x_n^2$, or else the determinant function on the real vector space of $n \times n$ symmetric, Hermitian, quaternion Hermitian, or octonion Hermitian matrices; in the octonion case, we set $n = 3$. Restrict f to the convex cone $\{f > 0, x_1 > 0\}$ in the quadratic form case, and to the cone of positive definite matrices in the other cases. Then the Hessian metric $-1/d(d-1)\partial^2 f/\partial x_i \partial x_j$ restricted to the hypersurface $M := \{f = 1\}$ in this convex cone gives a Riemannian symmetric space of noncompact type. In the quadratic form case, M is hyperbolic space of dimension $n-1$; in the other cases, we get the symmetric spaces $SL(n, \mathbf{R})/SO(n)$, $SL(n, \mathbf{C})/SU(n)$, $SL(n, \mathbf{H})/Sp(n)$, and E_6/F_4. A reference on symmetric cones and more general homogeneous convex cones is Vinberg [30].

3 Hessian metrics on $\mathbf{R}^3$ and the Clebsch covariant

Wilson gave a simple formula for the curvature of the Hessian metric associated to a cubic form f on $\mathbf{R}^3$ ([31], Theorem 5.1). (More precisely, he considered the curvature of the surface $\{f = 1\}$ in $\mathbf{R}^3$, but that is equivalent in view of Corollary 2.2.) In this section, we extend his formula to apply to a homogeneous function f of any degree greater than 2 on $\mathbf{R}^3$. The key ingredient in Wilson's formula is the classical Aronhold invariant S of a plane cubic curve; our formula involves a natural generalization, the Clebsch covariant.

For a cubic in Weierstrass form, the Aronhold invariant is $S = 2^2 3^{-3} g_2$; that is, S is a constant multiple of the Eisenstein series of weight 4. Explicitly, for any cubic form

$$f = a_3x_3^3+3(a_2x_1+b_2x_2)x_3^2+3(a_1x_1^2+2b_1x_1x_2+c_1x_2^2)x_3+(a_0x_1^3+3b_0x_1^2x_2+3c_0x_1x_2^2+d_0x_2^3),$$

the Aronhold invariant is given by

$$\begin{aligned} S = &-(a_0a_2-a_1^2)c_1^2+(a_0a_3-a_1a_2)c_0c_1-(a_1a_3-a_2^2)c_0^2-b_0^2a_3c_1+b_0b_1(3a_2c_1+a_3c_0) \\ &- (b_0b_2 + 2b_1^2)(a_1c_1 + a_2c_0) + b_1b_2(a_0c_1 + 3a_1c_0) - b_2^2a_0c_0 \\ &+ d_0[b_0(a_1a_3 - a_2^2) - b_1(a_0a_3 - a_1a_2) + b_2(a_0a_2 - a_1^2)] + (b_0b_2 - b_1^2)^2. \end{aligned}$$

Here I changed the sign of the Aronhold invariant from Elliott ([10], p. 377) to agree with the convention in Wilson's paper [31].

Clebsch observed that any invariant for forms of a given degree (in a given number of variables) extends in a natural way to a covariant for forms of any larger degree (in the same number of variables) [3]. One way to describe the construction is that we view the given invariant as an $SL(n)$-equivariant differential operator. For example, this procedure turns the discriminant invariant of quadratic forms into the Hessian covariant for forms of any degree.

This procedure turns the Aronhold invariant of cubic forms into the *Clebsch covariant* $S(f)$ for forms $f(x_1, x_2, x_3)$ of any degree, defined by Clebsch [3] and further studied by Dolgachev and Kanev [4]. Explicitly, $S(f)$ is defined by the same formula as the Aronhold invariant (above), but with a_i, b_i, c_i, d_i defined as:

$$\begin{aligned} &a_3 = f_{333} \\ &a_2 = f_{133} \quad b_2 = f_{233} \\ &a_1 = f_{113} \quad b_1 = f_{123} \quad c_1 = f_{223} \\ &a_0 = f_{111} \quad b_0 = f_{112} \quad c_0 = f_{122} \quad d_0 = f_{222}. \end{aligned}$$

Here f_{ijk} denotes the third derivative of f with respect to the variables x_i, x_j, x_k. This definition has the property that the Clebsch covariant $S(f)$ of a cubic form f is $2^4 3^4$ times the Aronhold invariant S of f. In general, the Clebsch covariant of a form f of degree d is a form of degree $4(d-3)$.

We now give our simplified formula for the curvature of the Hessian metric associated to a homogeneous function on $\mathbf{R}^3$. (To compare our formula with Wilson's formula on cubic forms f ([31], Theorem 5.1), one has to remember the above factor of $2^4 3^4$.)

Theorem 3.1 *Let f be a smooth homogeneous function of degree $d > 2$ on an open subset U of $\mathbf{R}^3$. Suppose that the Hessian determinant of f is nonzero on U, and consider the pseudo-Riemannian Hessian metric $-1/d(d-1)\partial^2 f/\partial x_i \partial x_j$ on U. Let $M = \{f = 1\}$. Then the sectional curvature of U on the tangent 2-plane to M at a point is given in terms of the Hessian determinant $H(f)$ and the Clebsch covariant $S(f)$ by*

$$K = \frac{d^2(d-1)^2}{4(d-2)^2} \frac{S(f)f}{H(f)^2}.$$

The sectional curvature of the surface M at the same point is

$$K_M = -\frac{d^2}{4} + \frac{d^2(d-1)^2}{4(d-2)^2} \frac{S(f)f^2}{H(f)^2}.$$

Notice that the theorem considers a point on M, thus with $f = 1$, and so the factors of f in the formulas are not strictly necessary. We include them so that the first formula gives, more generally, the sectional curvature of U on the tangent 2-plane to a point in any level set $\{f = \lambda\}$. This sectional curvature is homogeneous of degree $-d$ by Corollary 2.2. On the other hand, the formula for K_M includes f^2 so as to be homogeneous of degree 0: this formula gives, at any point x of U, the sectional curvature of the surface M at the point of M which is a scalar multiple of x.

Proof. By Corollary 2.2, the second formula follows from the first. So we just need to compute the sectional curvature of U at the tangent 2-plane to M at a point.

The equality we want is an algebraic identity among the derivatives of f. So it suffices to prove the same identity for holomorphic functions f on an open subset U of $\mathbf{C}^3$ which are homogeneous of degree d. The "metric" $-1/d(d-1)\partial^2 f/\partial x_i \partial x_j$ in this situation is a nondegenerate symmetric bilinear form on the tangent bundle to U. Sectional curvature is defined by the same formulas as for real metrics.

Both the sectional curvature of U at the tangent 2-plane to M and the right side of the above formula are unchanged under a change of coordinates in $GL(3, \mathbf{C})$. For the sectional curvature, this is clear. For the right side, it follows from the identities, for $A \in GL(3, \mathbf{C})$:

$$H(fA)|_x = H(f)|_{Ax} \det(A)^2$$
$$S(fA)|_x = S(f)|_{Ax} \det(A)^4.$$

So we can assume that the given point of M is $(0, 0, 1)$. Since this point is in M, we have $f(0, 0, 1) = 1$. By the Euler identity, it follows that $f_3|_{(0,0,1)} =$

$\sum x_i f_i|_{(0,0,1)} = d \cdot f(0,0,1) = d$. By a further change of coordinates in $GL(3,\mathbf{C})$, we can assume that the tangent plane to M at $(0,0,1)$ is spanned by the vectors $\partial/\partial x_1$ and $\partial/\partial x_2$. Equivalently, $\partial f/\partial x_1 = \partial f/\partial x_2 = 0$ at the point $(0,0,1)$. By the Euler identity again, at the point $(0,0,1)$ we have $f_{13} = f_{23} = 0$ and $f_{33} = d(d-1)$. Finally, we are assuming that the Hessian determinant of f is nonzero at the point $(0,0,1)$. So we can make one last change of coordinates in $GL(3,\mathbf{C})$ so as to make $f_{11} = d(d-1)$, $f_{12} = 0$, and $f_{22} = d(d-1)$ at the point $(0,0,1)$. (Over $\mathbf{R}$ we would have several cases, depending on the signature of the Hessian metric.)

By the Euler identity, we have $f_{113} = d(d-1)(d-2)$, $f_{123} = 0$, $f_{223} = d(d-1)(d-2)$, $f_{133} = 0$, $f_{233} = 0$, and $f_{333} = d(d-1)(d-2)$ at the point $(0,0,1)$. Also, the matrix $g_{ij} = -1/d(d-1)\partial^2 f/\partial x_i x_j$ is the matrix -1 at the point $(0,0,1)$, and so its inverse g^{ij} is also the matrix -1. We can now use the formula in section 2 for the curvature tensor of the Hessian metric $-1/d(d-1)\partial^2 f/\partial x_i \partial x_j$ on $\mathbf{C}^3$, applied to the 2-plane $T_{(0,0,1)}M$ spanned by $\partial/\partial x_1$ and $\partial/\partial x_2$:

$$\begin{aligned} R_{1212} &= -\frac{1}{4d^2(d-1)^2}\sum_{p,q} g^{pq}(f_{11q}f_{22p} - f_{12p}f_{12q}) \\ &= \frac{1}{4d^2(d-1)^2}\sum_{p}(f_{11p}f_{22p} - f_{12p}^2) \\ &= \frac{1}{4d^2(d-1)^2}(f_{111}f_{122} - f_{112}^2 + f_{112}f_{222} - f_{122}^2 + d^2(d-1)^2(d-2)^2). \end{aligned}$$

The sectional curvature of U at this 2-plane is $R_{1212}/(g_{11}g_{22} - g_{12}^2)$. This is equal to R_{1212}, and so the sectional curvature is given by the formula above.

We compare this to the Clebsch covariant of f at the same point $(0,0,1)$. Most terms vanish because of what we know about the third derivatives of f (in the notation used in defining the Clebsch covariant, $a_2 = f_{133}$, $b_2 = f_{233}$, and $b_1 = f_{123}$ are zero). What remains is:

$$\begin{aligned} S(f) &= a_1^2 c_1^2 + a_0 a_3 c_0 c_1 - a_1 a_3 c_0^2 - b_0^2 a_3 c_1 + d_0 b_0 a_1 a_3 \\ &= d^2(d-1)^2(d-2)^2(f_{111}f_{122} - f_{112}^2 + f_{112}f_{222} - f_{122}^2 + d^2(d-1)^2(d-2)^2). \end{aligned}$$

Therefore the curvature of U at the above 2-plane is $S(f)/(4d^4(d-1)^4(d-2)^2)$. At this point $(0,0,1)$, f is equal to 1 and the Hessian determinant of f is $d^3(d-1)^3$. So we can rewrite the curvature of U at the above 2-plane as

$$K = \frac{d^2(d-1)^2}{4(d-2)^2}\frac{S(f)f}{H(f)^2}.$$

This is the formula we want. QED

4 A quartic form on $\mathbf{R}^3$

In this section, we answer a question raised in the introduction to Wilson's paper, by giving the first example of a real form f such that the submanifold $M = \{f = 1\}$ of the index cone is nonempty and has positive curvature everywhere. Here M is given the Riemannian metric $-1/d(d-1)\partial^2 f/\partial x_i \partial x_j$, and the index cone is defined in section 1. The example here is a quartic form in $\mathbf{R}^3$, while the next section gives a cubic form on $\mathbf{R}^4$ with analogous properties. These examples are optimal: M has constant curvature -1 whenever f is a quadratic form with nonempty index cone, and M has at least one point of nonpositive curvature when f is a cubic form on $\mathbf{R}^3$ with nonempty index cone, by Wilson ([31], Prop. 5.2).

Lemma 4.1 *For the real quartic form $f = xyz(x+y+z)$, the index cone in $\mathbf{R}^3$ is nonempty, and the surface $M = \{f = 1\}$ inside the index cone has positive curvature everywhere.*

Note that the form f is not equivalent under $GL(3, \mathbf{R})$ to its negative, and indeed our argument will not apply to the negative of f.

Proof. The Hessian matrix of f is

$$(\partial^2 f/\partial x_i \partial x_j) = \begin{pmatrix} 2yz & 2xz+2yz+z^2 & 2xy+2yz+y^2 \\ 2xz+2yz+z^2 & 2xz & 2xy+2xz+x^2 \\ 2xy+2yz+y^2 & 2xy+2xz+x^2 & 2xy \end{pmatrix}.$$

We compute that the point $(1,1,1)$ is in the index cone; that is, f is positive at this point and the Hessian matrix has signature $(+,-,-)$ at this point. So the index cone is nonempty. More precisely, we compute that the Hessian of f (the determinant of the Hessian matrix) is

$$H(f) = 6xyz(x+y+z)(x^2+y^2+z^2+xy+xz+yz).$$

Here the quadratic form $x^2+y^2+z^2+xy+xz+yz$ on $\mathbf{R}^3$ is positive definite. Using that, it is straightforward to check that the index cone of f is equal to the "positive cone" $\{f > 0\}$.

We now show that the surface M, with the metric $-1/d(d-1)\partial^2 f/\partial x_i \partial x_j$, has positive curvature everywhere. By Theorem 3.1, for a quartic form f in 3 variables, the curvature of M has the form:

$$K = -4 + 9\frac{S(f)f^2}{H(f)^2}.$$

Thus, we need to show that $S(f) > 2^2 3^{-2} H(f)^2/f^2$ on the whole index cone. By the above calculation of the Hessian, it is equivalent to show that the quartic form

$$S(f) - 2^4(x^2+y^2+z^2+xy+xz+yz)^2$$

is positive on the whole index cone.

We compute that

$$S(f) = 2^4(x^4+2x^3y+3x^2y^2+2xy^3+y^4+2x^3z+7x^2yz+7xy^2z+2y^3z+3x^2z^2 \\ +3y^2z^2+7xyz^2+2xz^3+2yz^3+z^4).$$

Therefore the above difference is

$$\begin{aligned} S(f) - 2^4(x^2+y^2+z^2+xy+xz+yz)^2 &= 2^4 3xyz(x+y+z) \\ &= 2^4 3f. \end{aligned}$$

Since f is positive on the index cone, the above difference is positive on the index cone, as we want. QED

5 A cubic form on $\mathbf{R}^4$

This section gives another example which answers the question raised in Wilson's paper and discussed in section 4. The example here is a cubic form on $\mathbf{R}^4$ rather than a quartic form on $\mathbf{R}^3$.

Lemma 5.1 *For the real cubic form $f = (x_0^2 + x_1^2 - x_2^2 - x_3^2)x_3$, the index cone in $\mathbf{R}^4$ is nonempty, and the 3-manifold $M = \{f = 1\}$ inside the index cone has positive sectional curvature on some 2-plane at every point.*

Proof. The Hessian matrix of f is

$$\begin{pmatrix} 2x_3 & 0 & 0 & 2x_0 \\ 0 & 2x_3 & 0 & 2x_1 \\ 0 & 0 & -2x_3 & -2x_2 \\ 2x_0 & 2x_1 & -2x_2 & -6x_3 \end{pmatrix}.$$

The Hessian $H(f)$, the determinant of this matrix, is $2^4 x_3^2(x_0^2 + x_1^2 - x_2^2 + 3x_3^2)$. Therefore $H(f)$ is negative if and only if x_3 is not zero and $x_0^2 + x_1^2 - x_2^2 + 3x_3^2 < 0$. At these points, the Hessian matrix must have signature $(+,-,-,-)$ or $(+,+,+,-)$. By inspecting the upper left 2×2 submatrix of the Hessian matrix, we see that f has signature $(+,-,-,-)$ if and only if $x_3 < 0$ and $x_0^2 + x_1^2 - x_2^2 + 3x_3^2 < 0$. At such a point, we have

$$\begin{aligned} x_0^2 + x_1^2 - x_2^2 - x_3^2 &\leq x_0^2 + x_1^2 - x_2^2 + 3x_3^2 \\ &< 0, \end{aligned}$$

and so $f = (x_0^2 + x_1^2 - x_2^2 - x_3^2)x_3$ is positive. That is, the index cone of f is

$$\{(x_0, x_1, x_2, x_3) \in \mathbf{R}^4 : x_3 < 0,\ x_0^2 + x_1^2 - x_2^2 + 3x_3^2 < 0\}.$$

It is then easy to check that this index cone is nonempty (it has two connected components, each a half of a component of a standard circular cone in $\mathbf{R}^4$).

To compute the curvature of the 3-manifold $M = \{f = 1\}$ in the index cone, it is helpful to notice that the form f has a big automorphism group. In particular, the orthogonal group of the quadratic form $x_0^2 + x_1^2 - x_2^2$, fixing the coordinate x_3, preserves the form f. This group can move any point in $\mathbf{R}^4$ to a point with $x_0 = 0$. Since this group preserves the form f, it preserves the Hessian metric associated to f. Therefore it suffices to show that for every point of M such that $x_0 = 0$, the sectional curvature is positive on some 2-plane.

We can reduce to a lower-dimensional problem, using the symmetries of f. Namely, the group $\mathbf{Z}/2$, acting on $\mathbf{R}^4$ by changing the sign of x_0, preserves the form f, and so it preserves the metric on the 3-manifold $\{f = 1\}$ in the index cone. Therefore the fixed point set of this group action is a totally geodesic submanifold. Clearly this fixed point set is the surface $\{x_0 = 0, f = 1\}$ in the index cone. Therefore, the sectional curvature of this surface is equal to the sectional curvature of the 3-manifold at the corresponding 2-plane. Thus, it suffices to show that the surface $\{x_0 = 0, f = 1\}$ in the index cone has positive curvature everywhere.

It is clear that the restriction of our metric $-1/d(d-1)\partial^2 f/\partial x_i \partial x_j$ on M to this surface is the analogous metric associated to the form $f|_{x_0=0}$ on $\mathbf{R}^3$. So we need to show that the metric associated to the cubic form $g := (x_1^2 - x_2^2 - x_3^2)x_3$ on $\mathbf{R}^3$, restricted to the surface $\{g = 1\}$, has positive curvature at all points with $x_3 < 0$ and $x_1^2 - x_2^2 + 3x_3^2 < 0$.

We compute that the Hessian of g is $H(g) = 8x_3(x_1^2 - x_2^2 + 3x_3^2)$ and that the Clebsch invariant of g is $S(g) = 16$. (Recall that this paper's definition of the Clebsch invariant makes it, for plane cubics, equal to $2^4 3^4$ times the Aronhold invariant S, as used in Wilson's paper [31].) By Wilson's calculation for plane cubics ([31], Theorem 5.1), generalized in Theorem 3.1 in this paper, the curvature of the metric on $g = 1$ is:

$$\begin{aligned} K &= -\frac{9}{4} + 9\frac{S(g)g^2}{H(g)^2} \\ &= -2^{-2}3^2 + 2^{-2}3^2\frac{(x_1^2 - x_2^2 - x_3^2)^2}{(x_1^2 - x_2^2 + 3x_3^2)^2}. \end{aligned}$$

We are considering the region where $x_3 < 0$ and $x_1^2 - x_2^2 + 3x_3^2 < 0$. In this region, we have

$$x_1^2 - x_2^2 - x_3^2 < x_1^2 - x_2^2 + 3x_3^2 < 0.$$

So the above formula shows that the curvature of the surface $g = 1$ is positive everywhere in this region. QED

6 Hessian metrics of constant curvature

Wilson showed that for a Fermat form f of any degree d and any number of variables n, if the associated Hessian metric is Lorentzian (that is, of signature $(1, *)$) on a nonempty open subset of $\mathbf{R}^n$, then the associated Riemannian metric on $M = \{f = 1\}$ has constant sectional curvature $-d^2/4$ [31]. In this section, we describe a larger class of forms which give metrics of constant curvature $-d^2/4$ on M. Dubrovin showed that certain complex reflection groups (Shephard groups) give further examples of forms with this property. In 3 variables, Dubrovin's construction gives just one "new" example, the Maschke sextic. We ask whether the forms we find, together with Dubrovin's example, are the only forms in 3 variables which give metrics of curvature $-d^2/4$ (Question 6.2). We reformulate this as a problem in invariant theory (Question 6.3), which we study for the rest of the paper.

One justification for studying the condition that M has constant curvature $-d^2/4$ is that, at least when $n = 3$ and the form f is irreducible, this is the only possible constant value of the curvature, as one can deduce from Theorem 3.1. Another is that this condition is equivalent to flatness of the Hessian metric on $\mathbf{R}^n$, by Corollary 2.3.

In contrast to the previous sections about inequalities on the curvature, the Lorentzian condition does not play an important role here. We may as well ask the more general question: for which real forms f on $\mathbf{R}^n$ does the pseudo-Riemannian Hessian metric $-1/d(d-1)\partial^2 f/\partial x_i \partial x_j$ restricted to the hypersurface $M := \{f = 1\}$ have constant curvature $-d^2/4$ on a nonempty open subset of M? As in the proof of Theorem 3.1, it is convenient to work even more generally, with a holomorphic "metric" on a complex manifold, meaning a nondegenerate symmetric bilinear form on the tangent bundle. (Beware that this is not the usual kind of metric considered in complex differential geometry, which is Hermitian rather than bilinear.) We can ask: for which complex forms f on $\mathbf{C}^n$ does the Hessian metric $-1/d(d-1)\partial^2 f/\partial x_i \partial x_j$ restricted to the hypersurface $M := \{f = 1\}$ have constant curvature $-d^2/4$ on a nonempty open subset of M?

By Corollary 2.3, the hypersurface M has constant curvature $-d^2/4$ if and only if the Hessian metric is flat on a nonempty open subset of $\mathbf{C}^n$. So the above problem is equivalent to the more natural-looking problem of classifying forms f such that the Hessian metric is flat on a nonempty open subset of $\mathbf{C}^n$. Since the sectional curvature is defined by algebraic formulas (section 2), flatness on a nonempty open subset implies flatness on a dense open subset of $\mathbf{C}^n$.

Here is a simple class of flat Hessian metrics.

Lemma 6.1 *For any form f on $\mathbf{C}^2$, the Hessian metric associated to f is flat (wherever this makes sense, that is, where the Hessian determinant is nonzero).*

Proof. Since $M = \{f = 1\}$ is a complex 1-manifold, the restriction of the Hessian metric on $\mathbf{C}^2$ to M has constant curvature $-d^2/4$ on all complex 2-planes (this condition being vacuous). By Corollary 2.3, it follows that the Hessian metric on $\mathbf{C}^2$ is flat. QED

In fact, this proof shows how to construct an isometry from the Hessian metric associated to a real binary form f to a standard flat pseudo-Riemannian metric on $\mathbf{R}^2$. Assume, for example, that the Hessian of f has signature $(1, 1)$ on some open subset of $\mathbf{R}^2$ where $f > 0$. Then the metric we consider, $-1/d(d-1)\partial^2 f/\partial x_i \partial x_j$, also has signature $(1, 1)$, and its restriction to $M = \{f = 1\}$ is positive definite. Therefore we can construct an isometry from $(d^2/4)M$ to some open subset of the manifold $N = \{g = 1\}$, where $g(x, y) := x^2 - y^2$, since N is another 1-manifold with positive definite metric. By Lemma 2.1, this isometry extends to an isometry from $d^2/4$ times the Hessian metric of f on $\mathbf{R}^2$ to the Hessian metric of g on $\mathbf{R}^2$, which is the standard flat Lorentzian metric on $\mathbf{R}^2$.

From Lemma 6.1, it follows that any form on $\mathbf{C}^3$ which can be written $f = \alpha(x, y) + \beta(z)$ in some linear coordinate system gives a flat pseudo-Riemannian Hessian metric, wherever the Hessian determinant is nonzero. Indeed, the corresponding metric is the product of a flat metric on an open subset of $\mathbf{C}^2$, by Lemma 6.1, with a metric on $\mathbf{C}$, which is automatically flat. More generally, in any dimension, this argument shows that the Hessian metric of any form $f = \alpha_1(x_1, x_2) + \alpha_2(x_3, x_4) + \cdots$ is flat. This generalizes Wilson's observation ([31], Introduction, Example 2) that the real Fermat form $f = x_1^d - x_2^d - \cdots - x_n^d$ determines a metric on $M = \{f = 1\}$ with constant curvature $-d^2/4$, or equivalently a flat metric on an open subset of $\mathbf{R}^n$ by Corollary 2.3. In $\mathbf{C}^3$, we can ask if the forms we have found, together with Dubrovin's example of the Maschke sextic (([7], Corollary 5.9 and Example 3), are essentially the only ones for which the Hessian metric is flat:

Question 6.2 *Let f be a ternary form of degree d over $\mathbf{C}$ whose Hessian determinant is not identically zero. Suppose that the Hessian metric $-1/d(d-1)\partial^2 f/\partial x_i \partial x_j$ is flat on a nonempty open subset of $\mathbf{C}^3$. Is f either in the closure of the set of forms which can be written as $\alpha(x, y) + \beta(z)$ in some linear coordinates, or equal to the Maschke sextic $x^6 + y^6 + z^6 - 10(x^3y^3 + y^3z^3 + z^3x^3)$ in some linear coordinates?*

By Theorem 3.1, Question 6.2 is equivalent to the following question. This question is formally more general in that it makes sense even for forms f whose Hessian determinant is identically zero. (Gordan and Noether showed that a ternary form whose Hessian determinant $H(f)$ is identically zero can be written as $\alpha(x, y)$ in some linear coordinates (x, y, z) [12].) The question could have been asked by the 19th-century invariant theorists. The rest of the paper will be devoted to it.

Question 6.3 *Let f be a ternary form of degree d over $\mathbf{C}$ whose Clebsch covariant $S(f)$ is identically zero. Is f either in the closure of the set of forms which can be written as $\alpha(x, y) + \beta(z)$ in some linear coordinates, or equal to the Maschke sextic $x^6 + y^6 + z^6 - 10(x^3y^3 + y^3z^3 + z^3x^3)$ in some linear coordinates?*

7 The closure of the set of forms $\alpha(x, y) + \beta(z)$

To clarify Question 6.3, in this section we give an explicit description of the closure of the set of ternary forms over $\mathbf{C}$ which can be written as $\alpha(x, y) + \beta(z)$ in some linear coordinate system.

Lemma 7.1 *Let f be a ternary form over $\mathbf{C}$. Then f is in the closure of the set of forms which can be written as $\alpha(x, y) + \beta(z)$ in some linear coordinates if and only if f can be written as either $\alpha(x, y) + \beta(z)$ or $\alpha(x, y) + \beta(x)z$ in some linear coordinates.*

Proof. It is easy to show that the second condition implies the first. That is, we have to show that any form $f = \alpha(x, y) + \beta(x)z = \alpha(x, y) + bx^{d-1}z$ is in the closure of the set of $GL(3, \mathbf{C})$-translates of forms $\gamma(x, y) + \delta(z)$. This is immediate by noting that

$$f = \lim_{c \to 0} \left[\left(-\frac{x^d}{c} + \alpha(x, y) \right) + \frac{1}{c} \left(x + \frac{cb}{d} z \right)^d \right].$$

For the converse, we have to show that any ternary form in the closure of the set of $GL(3, \mathbf{C})$-translates of forms $\alpha(x, y) + \beta(z)$ can be written either as $\alpha(x, y) + \beta(z)$ or as $\alpha(x, y) + \beta(x)z$, in some linear coordinates.

We use that any form $f = \alpha(x, y) + \beta(z)$ satisfies the differential equations

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial y \partial z} = 0.$$

Thus, in the 3-dimensional complex vector space spanned by $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, there is a 1-dimensional subspace L and a 2-dimensional subspace P such that $LPf = 0$. Clearly this remains true for any $GL(3, \mathbf{C})$-translate of f. Therefore it remains true for any form g in the closure of such forms, although the line L may be contained in the plane P. Thus, after a change of coordinates, g either satisfies

$$\frac{\partial^2 g}{\partial x \partial z} = \frac{\partial^2 g}{\partial y \partial z} = 0$$

or

$$\frac{\partial^2 g}{\partial z^2} = \frac{\partial^2 g}{\partial y \partial z} = 0.$$

In the first case we can write $g = \alpha(x, y) + \beta(z)$, and in the second case we can write $g = \alpha(x, y) + \beta(x)z$. QED

8 Plane quartic curves with vanishing Clebsch covariant

In this section, we give a positive answer to Question 6.3 for forms of degree at most 4.

Theorem 8.1 *A ternary form f of degree at most 4 over the complex numbers has Clebsch covariant equal to zero if and only if it is in the closure of the set of forms which can be written as $\alpha(x, y) + \beta(z)$ in some linear coordinates.*

Proof. This is trivial for forms of degree at most 2 (where every form can be written as $\alpha(x, y) + \beta(z)$). For cubic forms, the Clebsch covariant is a constant multiple of the Aronhold invariant S, and it is a standard fact from the theory of elliptic curves that a cubic curve has $S = 0$ if and only if it is in the closure of the set of $GL(3, \mathbf{C})$-translates of the Fermat cubic $x^3 + y^3 + z^3$ ([4], Prop. 5.13.2).

It remains to show that a ternary quartic with Clebsch covariant zero is in the closure of the set of $GL(3, \mathbf{C})$-translates of the forms $\alpha(x, y) + \beta(z)$. (These forms are generally not in the closure of the orbit of the Fermat quartic $x^4 + y^4 + z^4$.)

Dolgachev and Kanev showed that a ternary quartic f with Clebsch covariant zero is not "weakly nondegenerate", in their terminology ([4], Cor. 6.6.3(iv)). That is, in some linear coordinates, we have either $\partial^2 f/\partial y \partial z = 0$ or $\partial^2 f/\partial y^2 = 0$. We are trying to prove a stronger conclusion with the same hypothesis, building on their result.

Suppose we are in the first case, that is, that $\partial^2 f/\partial y \partial z = 0$. Then we can write $f = \alpha(x, y) + \beta(x, z)$. To say more, we need to see what the vanishing of the Clebsch covariant says about such a form. In the notation used to define the Clebsch covariant, the forms $b_1 = f_{123}$, $c_1 = f_{223}$, and $b_2 = f_{233}$ are zero. So the formula for the Clebsch covariant becomes:

$$S(f) = (a_1 a_3 - a_2^2)(d_0 b_0 - c_0^2).$$

Thus, since the Clebsch covariant $S(f)$ is zero, we have either $a_1 a_3 = a_2^2$ or $d_0 b_0 = c_0^2$. These two conditions are the same up to switching the coordinates y and z, so we can assume that $a_1 a_3 = a_2^2$, that is, $f_{113} f_{333} = f_{133}^2$.

Since we have $f = \alpha(x, y) + \beta(x, z)$, the derivative f_3 is equal to β_3, and so the above equation says that the Hessian determinant of $\beta_3 = (\partial\beta/\partial z)(x, z)$ is zero. It is classical that the vanishing of the Hessian of a binary form implies that the form is a power of a linear form ([24], Prop. 2.23). Thus we can write $\beta_3 = (bx + cz)^{d-1}$ for some numbers b, c. If $c = 0$, so that β_3 is a constant multiple of x^{d-1}, then β itself is a linear combination of $x^{d-1}z$ and x^d. Thus, putting the x^d term into α, we can write the form f as $\alpha(x, y) + ax^{d-1}z$ for some a. This proves what we want in the case $c = 0$, since the forms $\alpha(x, y) + ax^{d-1}z$ are in the closure of the set of $GL(3, \mathbf{C})$-translates of the forms $\alpha(x, y) + \beta(z)$ by Lemma 7.1. It remains to consider the case where $\beta_3 = (bx + cz)^{d-1}$ with c not zero. Then β itself is a linear combination of $(bx + cz)^d$ and x^d. Putting the x^d term into α, we can write the form f as $\alpha(x, y) + a(bx + cz)^d$ for some a. Since c is not zero, we can change coordinates to write $f = \alpha(x, y) + \beta(z)$, as we want.

It remains to consider the second case of Dolgachev and Kanev's result, where we have $\partial^2 f/\partial y^2 = 0$. Intuitively, this should be a more special case than the previous one. In this case, we can write $f = \alpha(x,z) + \beta(x,z)y$. We need to work out what the vanishing of the Clebsch covariant tells us about such a form. In the notation defining the Clebsch covariant, the forms $c_0 = f_{122}$, $c_1 = f_{223}$, and $d_0 = f_{222}$ are zero. Therefore the formula for the Clebsch covariant simplifies to:

$$S(f) = (b_0 b_2 - b_1^2)^2.$$

Since the Clebsch covariant $S(f)$ is zero, we have $b_0 b_2 = b_1^2$, that is, $f_{211} f_{233} = f_{213}^2$. Here $f_2 = \beta(x,z)$, and so this equation says that the binary form $\beta(x,z)$ has Hessian equal to zero. Therefore $\beta(x,z)$ is a power of a linear form. Thus, after a linear change of coordinates in x and z, we can write $f = \alpha(x,z) + ax^{d-1}y$ for some a. This is the conclusion we want, since such forms are in the closure of the set of $GL(3,\mathbf{C})$-translates of forms $\alpha(x,y) + \beta(z)$ by Lemma 7.1. QED

9 An irreducible component of the set of plane curves with vanishing Clebsch covariant

In this section, we give further evidence for a positive answer to Question 6.3. Namely, we show that the forms which can be written as $\alpha(x,y) + \beta(z)$ in some linear coordinates constitute an irreducible component of the set of all forms with vanishing Clebsch covariant, for forms of any degree.

Theorem 9.1 *For any number d, let Y be the closure of the set of ternary forms f of degree d over the complex numbers which can be written as $\alpha(x,y) + \beta(z)$ in some linear coordinates. Then Y is an irreducible component of the set of forms f with Clebsch covariant $S(f) = 0$. Moreover, the scheme defined by $S(f) = 0$ is reduced at a general point of Y.*

Proof. For d at most 2, every ternary form of degree d belongs to Y, and the Clebsch covariant is identically zero; so the statement is clear. For $d = 3$, we know the stronger statement (Theorem 8.1) that the equation $S(f) = 0$ defines the set Y. Also, $S(f)$ is an irreducible invariant on the space of plane cubic curves, and so it defines Y as a reduced hypersurface. So we can assume that d is at least 4.

It is clear that the set Y is irreducible. We will show that there is a point of Y in a neighborhood of which the equation $S(f) = 0$ defines Y as a smooth scheme. This implies the same statement on a dense open subset of Y, and hence implies both statements of the theorem.

Clearly, a general point f of Y can be written, in some linear coordinates, as $f = \alpha(x,y) + \beta(z)$, for some forms $\alpha(x,y)$ and $\beta(z)$ of degree d. Thus we need to choose forms $\alpha(x,y)$ and $\beta(z)$ such that the equation $S(f) = 0$ defines Y as a smooth scheme in a neighborhood of the point f. We will take $\beta(z) = z^d$. (This is no loss of generality, since a general form $\beta(z) = cz^d$ can be put into this form by scaling z.)

There is an easy lower bound for the Zariski tangent space of Y at the point f. First, we can vary the forms $\alpha(x,y)$ and $\beta(z)$; and then, we can move $f = \alpha(x,y) + \beta(z)$ by elements of $GL(3,\mathbf{C})$, in the direction of an element of the Lie algebra $\mathfrak{gl}(3,\mathbf{C})$. The Lie subalgebra $\mathfrak{gl}(2) \times \mathfrak{gl}(1)$ maps the vector space of forms $\gamma(x,y) + \delta(z)$ into itself, and so it suffices to consider the 4-dimensional vector space $\mathfrak{g}l(3)/\mathfrak{g}l(2) \times \mathfrak{g}l(1)$. It is spanned by the four infinitesimal transformations $x \mapsto x + \epsilon z$, $y \mapsto y$, $z \mapsto z$; and likewise $y \mapsto y + \epsilon z$ with other variables unchanged; and likewise $z \mapsto z + \epsilon x$; and likewise $z \mapsto z + \epsilon y$. These transformations move the form $f = \alpha(x, y) + z^d$ in the directions of the following 4 forms: $\alpha_1 z$, $\alpha_2 z$, xz^{d-1}, yz^{d-1}. (Following section 1, α_1 means $\partial\alpha/\partial x$ and α_2 means $\partial\alpha/\partial y$ here.) Thus the Zariski tangent space to Y at the point f is at least the span of the forms $x^d, x^{d-1}y, \ldots, y^d$, $\alpha_1 z$, $\alpha_2 z$, xz^{d-1}, yz^{d-1}, z^d. Clearly, for a general form $\alpha(x, y)$ of degree d, the forms listed are linearly independent.

The theorem will be proved if we can show that there is a form $\alpha(x, y)$ of degree d such that the Zariski tangent space to the scheme $S(f) = 0$ at the point $f = \alpha(x, y) + z^d$ is spanned by the above forms. To compute the Zariski tangent space to the scheme $S(f) = 0$, we need to compute $S(f + \epsilon g) \pmod{\epsilon^2}$ for an arbitrary form $g(x, y, z)$, where $f = \alpha(x, y) + z^d$. From the definition of the Clebsch covariant, we compute:

$$\begin{aligned} S(\alpha(x,y) + z^d + \epsilon g) = \epsilon d^4 (d-1)^4 (d-2)^4 [&g_{113}(\alpha_{112}\alpha_{222} - \alpha_{122}^2) \\ &+ g_{123}(\alpha_{112}\alpha_{122} - \alpha_{111}\alpha_{222}) + g_{223}(\alpha_{111}\alpha_{122} - \alpha_{112}^2)] \pmod{\epsilon^2}. \end{aligned}$$

Thus, we need to show that there is a binary form $\alpha(x, y)$ of degree d such that the vanishing of the expression in brackets for a ternary form $g(x, y, z)$ of degree d implies that g is in the span of the forms $x^d, x^{d-1}y, \ldots, y^d$, $\alpha_1 z$, $\alpha_2 z$, xz^{d-1}, yz^{d-1}, z^d.

It is convenient to observe that the expression in brackets sends the part of the polynomial g such that z has a given exponent a to a polynomial such that z has exponent $a - 1$. As a result, we can consider the problem separately for each part of g of the form $h(x, y)z^a$. Thus the question becomes one about binary forms only. Namely, for binary forms $\alpha(x, y)$ and $h(x, y)$, define

$$T(\alpha, h) = h_{11}(\alpha_{112}\alpha_{222} - \alpha_{122}^2) + h_{12}(\alpha_{112}\alpha_{122} - \alpha_{111}\alpha_{222}) + h_{22}(\alpha_{111}\alpha_{122} - \alpha_{112}^2).$$

This is an $SL(2)$-equivariant differential operator in α and h. We need to show that for any $d \geq 4$, there is a form $\alpha(x, y)$ of degree d with the following properties. First, for any form $h(x, y)$ of degree r with $2 \leq r \leq d - 2$, if $T(\alpha, h) = 0$ then $h = 0$. Second, any form $h(x, y)$ of degree $d - 1$ with $T(\alpha, h) = 0$ must be a linear combination of the derivatives α_1 and α_2.

In fact, we can prove even more, as follows. This will complete the proof of Theorem 9.1. It would be preferable to have a more geometric interpretation of the operator $T(\alpha, h)$, but it happens that we can get by without that.

Lemma 9.2 *Let $\alpha(x, y)$ be a very general binary form of degree $d \geq 4$ (that is, a form outside countably many proper subvarieties of the space of all forms of degree d). Then the binary forms $h(x, y)$ of any degree such that $T(\alpha, h) = 0$ are the linear combinations of: 1 in degree 0, x and y in degree 1, the derivatives α_1 and α_2 in degree $d - 1$, and α in degree d.*

Proof. First, it is a straightforward calculation that for any form $\alpha(x, y)$, the operator $T(\alpha, h)$ vanishes when h has degree at most 1, and also that $T(\alpha, \alpha_1) = T(\alpha, \alpha_2) = T(\alpha, \alpha) = 0$. The operator $T(\alpha, h)$ is linear in h, and so $T(\alpha, h)$ vanishes when h is any linear combination of the forms $1, x, y, \alpha_1, \alpha_2, \alpha$.

To prove the lemma for a very general form $\alpha(x,y)$, it suffices to prove it for a single form $\alpha(x,y)$ of degree d. (We get the conclusion only for α very general, that is, outside countably many proper subvarieties, because the statement involves forms h of arbitrary degree. For the application to Theorem 9.1, we only need Lemma 9.2 for forms h of degree at most $d-1$, and that weakened form of the Lemma holds for forms α outside only finitely many proper subvarieties.)

We take $\alpha(x,y) = \binom{d}{2}x^{d-2}y^2$. Then, for any form h, we compute that

$$T(\alpha,h) = d^2(d-1)^2(d-2)^2x^{2d-8}[x^2h_{11} - (d-3)xyh_{12} + \frac{1}{2}(d-2)(d-3)y^2h_{22}].$$

Clearly a form h has $T(\alpha,h)$ equal to zero if and only if the expression $U(\alpha,h)$ in brackets is zero. We compute that

$$U(\alpha, x^iy^j) = x^iy^j(i(i-1) - (d-3)ij + \frac{1}{2}(d-2)(d-3)j(j-1)).$$

Thus the differential operator $h \mapsto U(\alpha,h)$ is diagonalized on the basis of monomials x^iy^j. It follows that the vector space of forms h with $T(\alpha,h)$ equal to zero is spanned by the set of monomials x^iy^j such that

$$i(i-1) - (d-3)ij + \frac{1}{2}(d-2)(d-3)j(j-1) = 0.$$

For fixed j, let us view this equation as a quadratic equation for i. As such, its discriminant $b^2 - 4ac$ is

$$\Delta = 1 - (d-1)(d-3)j(j-2).$$

Since d is at least 4, $(d-1)(d-3)$ is at least 3. So we read off that Δ is negative for j at least 3. Thus, there are no integral (or even real) solutions i unless $j \leq 2$. Solving our quadratic equation for $j = 0,1,2$ gives that the only solutions (i,j) in natural numbers are: $(0,0)$, $(1,0)$, $(0,1)$, $(d-2,1)$, $(d-3,2)$, and $(d-2,2)$. Thus, for $\alpha = \binom{d}{2}x^{d-2}y^2$ as we have been considering, the vector space of forms h such that $T(\alpha,h) = 0$ is spanned by $1, x, y, x^{d-2}y, x^{d-3}y^2$, and $x^{d-2}y^2$; equivalently, it is spanned by $1, x, y, \alpha_1, \alpha_2$, and α. This proves the lemma for the particular form $\alpha = \binom{d}{2}x^{d-2}y^2$. As we have said, this implies the lemma for very general forms α. QED

References

[1] S. Y. Cheng and S. T. Yau. On the regularity of the Monge-Ampère equation $\det(\partial^2 u/\partial x_i \partial x_j) = F(x,u)$. *Comm. Pure Appl. Math.* **30** (1977), 41–68.

[2] S. Y. Cheng and S. T. Yau. The real Monge-Ampère equation and affine flat structures. *Proc. 1980 Beijing Symposium on Differential Geometry and Differential Equations,* v. 1, 339-370 (1982).

[3] A. Clebsch. Über Curven vierter Ordnung. *J. Reine Angew. Math.* **59** (1861), 125–145.

[4] I. Dolgachev and V. Kanev. Polar covariants of plane cubics and quartics. *Adv. Math.* **98** (1993), 216–301.

[5] B. Dubrovin. Geometry of 2D topological field theories. *Integrable systems and quantum groups* (Montecatini Terme, 1993), 120–348, LNM 1620, Springer (1996).

[6] B. Dubrovin. Painlevé transcendents in two-dimensional topological field theory. *The Painlevé property* (Cargèse, 1996), 287–412, Springer (1999).

[7] B. Dubrovin. On almost duality for Frobenius manifolds. arXiv.org/math.DG/0307374

[8] J. Duistermaat. On the boundary behaviour of the Riemannian structure of a self-concordant barrier function. *Asymptot. Anal.* **27** (2001), 9–46.

[9] J. Duistermaat. On Hessian Riemannian structures. *Asian J. Math.* **5** (2001), 79–91.

[10] E. Elliott. *An introduction to the algebra of quantics.* Oxford (1895).

[11] E. Ferapontov. Hypersurfaces with flat centroaffine metric and equations of associativity. *Geom. Dedicata,* to appear.

[12] P. Gordan and M. Noether. Über die algebraischen Formen, deren Hesse'sche Determinante identisch verschwindet. *Math. Ann.* **10** (1876), 547–568.

[13] O. Güler. Barrier functions in interior point methods. *Math. Oper. Res.* **21** (1996), 860–885.

[14] O. Güler. Hyperbolic polynomials and interior point methods for convex programming. *Math. Oper. Res.* **22** (1997), 350–377.

[15] C. Hertling. *Frobenius manifolds and moduli spaces for singularities.* Cambridge (2002).

[16] N. Hitchin. The moduli space of special Lagrangian submanifolds. *Ann. Scuola Norm. Sup. Pisa* **25** (1997), 503–515.

[17] H. Kito. On Hessian structures on the Euclidean space and the hyperbolic space. *Osaka J. Math.* **36** (1999), 51–62.

[18] M. Kontsevich and Y. Soibelman. Homological mirror symmetry and torus fibrations. *Symplectic geometry and mirror symmetry* (Seoul, 2000), 203–263, World Scientific (2001).

[19] N. C. Leung. Mirror symmetry without corrections. *Comm. Anal. Geom.,* to appear.

[20] C. Loewner and L. Nirenberg. Partial differential equations invariant under conformal or projective transformations. *Contributions to analysis,* Academic Press (1974), 245–272.

[21] J. Loftin. Affine spheres and Kähler-Einstein metrics. *Math. Res. Lett.* **9** (2002), 425–432.

[22] J. Moore and J.-M. Morvan. On isometric Lagrangian immersions. *Illinois J. Math.* **45** (2001), 833–849.

[23] C. Okonek and A. van de Ven. Cubic forms and complex 3-folds. *Ens. Math.* **41** (1995), 297–333.

[24] P. Olver. *Classical invariant theory.* Cambridge (1999).

[25] B. O'Neill. *Semi-Riemannian geometry.* Academic Press (1983).

[26] T. Sasaki. Hyperbolic affine hyperspheres. *Nagoya Math. J.* **77** (1980), 107–123.

[27] H. Shima and K. Yagi. Geometry of Hessian manifolds. *Differential Geom. Appl.* **7** (1997), 277–290.

[28] M. Spivak. *A comprehensive introduction to differential geometry,* v. 2. Publish or Perish (1979).

[29] C.-L. Terng. Geometries and symmetries of soliton equations and integrable elliptic equations. *Surveys on geometry and integrable systems,* Math. Soc. Japan, to appear.

[30] E. Vinberg. The theory of convex homogeneous cones. *Trans. Moscow Math. Soc.* **12** (1963), 340–403.

[31] P.M.H. Wilson. Sectional curvatures of Kähler moduli. arXiv.org/math.AG/0307260

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