Towards a Schubert calculus for complex reflection groups

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Schubert in 1886 found a beautiful formula for the degrees of Grassmannians and their Schubert subvarieties, viewed as projective algebraic varieties. See [16] or [5], example 14.7.11. Combined with a formula of Giambelli's ([7] or [5], example 14.5.1), Schubert's formula also gives the degree of the isotropic Grassmannian $Sp(2n)/U(n)$:

$$2^{\binom{n}{2}} \frac{\binom{n+1}{2}!\, 1!\, 2! \cdots (n-1)!}{1!\, 3! \cdots (2n-1)!}.$$

Geometrically, this is the number of Lagrangian subspaces of a $2n$-dimensional symplectic vector space which have nonzero intersection with $\binom{n+1}{2}$ general Lagrangian subspaces.

Recently, there have been various attempts to generalize known results on Lie groups, when they can be expressed in terms of the Weyl group, to complex reflection groups. See Broué-Malle-Rouquier [4], Broué-Malle-Michel [3], Bremke-Malle [1], [2], Rampetas-Shoji [15], and Rampetas [14]. In this spirit, we will generalize the above formula by replacing the symplectic group $Sp(2n)$, or rather its Weyl group (the wreath product $S_n \wr \mathbf{Z}/2$ acting on $\mathbf{R}^n$), by the complex reflection group $G(e,1,n) = S_n \wr \mathbf{Z}/e$ acting on $\mathbf{C}^n$. The generalization no longer has any obvious geometric meaning, but experience suggests that this kind of algebra will eventually find a geometric or representation-theoretic interpretation. Besides generalizing the above degree formula, we encounter a generalization of the classical relation between the cohomology rings of $Sp(2n)/U(n)$ and $SO(2n+1)/U(n)$.

Our generalization of the above degree formula involves the ring

$$C(e,n) := \mathbf{Z}[e_1, \ldots, e_n]/(e_i(x_1^e, \ldots, x_n^e), i \geq 1),$$

where e_i denotes the ith elementary symmetric function in variables $x_1, \ldots, x_n$. For $e = 2$, $C(2,n)$ is the integral cohomology ring of the isotropic Grassmannian $Sp(2n)/U(n)$. The degree of the isotropic Grassmannian can be defined in terms of this cohomology ring: the top-dimensional group $\mathbf{Z}$ is generated by $e_1 \cdots e_n$, and the degree of the isotropic Grassmannian is the coefficient of $e_1^{\binom{n+1}{2}}$ as a multiple of $e_1 \cdots e_n$. The main result of this paper is a calculation of the analogous number for the ring $C(e,n)$ for arbitrary positive integers e and n. This ring satisfies Poincaré duality over the integers, with top-dimensional group $\mathbf{Z}$ generated by $(e_1 \cdots e_n)^{e-1}$.

Theorem 0.1 *In the ring $C(e,n)$, the coefficient of $e_1^{\binom{n+1}{2}(e-1)}$ as a multiple of the element $(e_1 \cdots e_n)^{e-1}$ is*

$$e^{\binom{n}{2}} \frac{(\binom{n+1}{2}(e-1))!\, 1!\, 2! \cdots (n-1)!}{(e-1)!\, (2e-1)! \cdots (ne-1)!}.$$

The above number appears in the proof as the number of standard tableaux of shape equal to the partition $\lambda = (n(e-1), (n-1)(e-1), \ldots, e-1)$ of $N = \binom{n+1}{2}(e-1)$. The number of such tableaux is called the Kostka number $K_{\lambda,(1^N)}$; it is the dimension of the irreducible representation of the symmetric group S_N corresponding to the partition λ [10].

In the case $e = 2$, the degree of the isotropic Grassmannian can be computed using the Schubert basis for the ring $C(2, n)$ together with the Pieri formula which computes the product of any basis element with e_1. This approach in fact gives a better combinatorial interpretation of the above number for $e = 2$: it is $2^{\binom{n}{2}}$ times the number of 'shifted standard tableaux' of shape $(n, n-1, \ldots, 1)$ [10]. The number of shifted standard tableaux of this shape (without the factor $2^{\binom{n}{2}}$) also arises as the degree of the spinor variety $SO(2n+1)/U(n)$ in its basic projective embedding. For general e and n, we can use Hall-Littlewood symmetric functions, specialized to an eth root of unity ζ, to give a basis for the ring $C(e, n)$ (tensored with $\mathbf{Z}[\zeta]$). We can also use Hall-Littlewood functions to define a related algebra $B(e, n)$ which gives the integral cohomology ring of $SO(2n+1)/U(n)$ in the case $e = 2$. By comparing the two rings, we find another interpretation for the number in our main formula, generalizing the shifted standard tableaux which come up for $e = 2$, but now involving a sum of roots of unity.

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1 Proof of Theorem 0.1

We want to compute the degree of $e_1^{\binom{n+1}{2}(e-1)}$, meaning the integer d such that

$$e_1^{\binom{n+1}{2}(e-1)} = d(e_1 \cdots e_n)^{e-1}$$

in the ring

$$C(e, n) := \mathbf{Z}[e_1, \ldots, e_n]/(e_i(x_1^e, \ldots, x_n^e), i \geq 1).$$

We view $C(e, n)$ as a subring of the ring

$$F(e, n) := \mathbf{Z}[x_1, \ldots, x_n]/(e_i(x_1^e, \ldots, x_n^e), i \geq 1).$$

For $e = 2$, the inclusion $C(e, n) \subset F(e, n)$ is the inclusion of the cohomology of the isotropic Grassmannian $Sp(2n)/U(n)$ in that of the isotropic flag manifold $Sp(2n)/T$, where $T = (S^1)^n$ is a maximal torus in the symplectic group. For $e \geq 3$, there is no Lie group whose Weyl group is the complex reflection group $S_n \wr \mathbf{Z}/e$, but it is helpful to think of the ring $F(e, n)$ as what the cohomology of the flag manifold associated to such a Lie group would be if one existed.

Define an element of dimension $\binom{n}{2}$ in $F(e, n)$ by the Vandermonde determinant,

$$\begin{aligned} \Delta &= \prod_{i<j} (x_i - x_j) \\ &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) x_1^{\sigma(n)-1} \cdots x_n^{\sigma(1)-1}. \end{aligned}$$

The symmetric group S_n acts on $F(e, n)$ by permuting the x_i's, and Δ is antisymmetric under this action. This element comes up naturally in the study of flag manifolds as the lowest-dimensional element of $F(e, n)$ on which the symmetric group S_n acts by the sign representation. Furthermore, multiplication by Δ gives an injection (in fact, multiplication by $n!$) from the group $\mathbf{Z}$ in the top dimension, $\binom{n+1}{2}(e-1)$, of $C(e, n)$ to the group $\mathbf{Z}$ in the top dimension, $\binom{n+1}{2}e - n$, of $F(e, n)$. That is not hard to check directly, but we will show it by explicitly computing $(e_1 \cdots e_n)^{e-1}\Delta$ and $e_1^{\binom{n+1}{2}(e-1)}\Delta$ and seeing that they are nonzero in the top-dimensional group $\mathbf{Z}$ of $F(e, n)$. Then we can compute the degree of $e_1^{\binom{n+1}{2}(e-1)}$ in $C(e, n)$ as the coefficient of the second element divided by that of the first.

We begin with some lemmas which we will use to compute in the top dimension of the ring $F(e, n)$.

Lemma 1.1 *In the ring $F(e,n)$, we have the relations*

$$\begin{aligned} x_n^{ne} &= 0 \\ x_{n-1}^{(n-1)e} x_n^{(n-1)e} &= 0 \\ &\vdots \\ x_1^e \cdots x_n^e &= 0. \end{aligned}$$

Proof. Let us define an S_n-equivariant homomorphism α from the ring

$$H^*(U(n)/T, \mathbf{Z}) = \mathbf{Z}[x_1, \ldots, x_n]/(e_i(x_1, \ldots, x_n) = 0, i \geq 1)$$

to $F(e,n)$, multiplying dimensions by e, by sending x_i to x_i^e for $i = 1, \ldots, n$. The homomorphism α implies the desired relations in $F(e,n)$ if we can prove the relations

$$\begin{aligned} x_n^n &= 0 \\ x_{n-1}^{n-1} x_n^{n-1} &= 0 \\ &\vdots \\ x_1 \cdots x_n &= 0 \end{aligned}$$

in $H^*(U(n)/T, \mathbf{Z})$. We can prove these relations by a simple geometric argument, although it could be reformulated algebraically. Namely, the cohomology class $x_{n+1-i} \cdots x_n$ in H^{2i} of the flag manifold $U(n)/T$ is pulled back from the Grassmannian $U(n)/(U(i) \times U(n-i))$, which has complex dimension $i(n-i)$. So raising that class to the power $n-i+1$ gives 0. That proves the desired relations. QED

Lemma 1.2 *Any monomial in the generators $x_1, \ldots, x_n$ of dimension $\binom{n+1}{2}e - n$ (the top dimension of the ring $F(e,n)$) is equal to 0 in the ring $F(e,n)$ unless it has the form $x_1^{\sigma(n)e-1} \cdots x_n^{\sigma(1)e-1}$ for some σ in the symmetric group S_n. In that case, such a monomial is equal to $sgn(\sigma) x_1^{ne-1} \cdots x_{n-1}^{2e-1} x_n^{e-1}$ in $F(e,n)$.*

Proof. We first observe that all permutations of the relations given in Lemma 1.1 are also valid in $F(e,n)$, since the symmetric group S_n acts on the ring $F(e,n)$. It follows that any monomial in $x_1, \ldots, x_n$ which is nonzero in $F(e,n)$ must have no exponent $\geq ne$, at most 1 exponent $\geq (n-1)e$, $\ldots$, and at most $n-1$ exponents $\geq e$. So every monomial in $x_1, \ldots, x_n$ of dimension $\binom{n+1}{2}e - n$ which is nonzero must have the form $x_1^{\sigma(n)e-1} \cdots x_n^{\sigma(1)e-1}$ for some σ in S_n.

To show that such a monomial is equal to $\mathrm{sgn}(\sigma)$ times $x_1^{ne-1} \cdots x_{n-1}^{2e-1} x_n^{e-1}$, it suffices to show that the symmetric group S_n acts by the sign representation on the top-dimensional group, isomorphic to $\mathbf{Z}$, of the ring $F(e,n)$. This follows from the analogous fact, which is well known, for the cohomology of the classical flag manifold $U(n)/T$. To use that, we will apply the S_n-equivariant homomorphism

$$\alpha : H^*(U(n)/T, \mathbf{Z}) \to F(e,n)$$

from the proof of Lemma 1.1. We know that S_n acts by the sign representation on the top-dimensional basis element $x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$ of $H^*(U(n)/T, \mathbf{Z})$, so S_n acts by the sign representation on its image $x_1^{(n-1)e} x_2^{(n-2)e} \cdots x_{n-1}^e$ in $F(e,n)$. Since S_n acts trivially on the element $(x_1 \cdots x_n)^{e-1}$, it acts by the sign representation on the product

$$(x_1^{(n-1)e} x_2^{(n-2)e} \cdots x_{n-1}^e)(x_1 \cdots x_n)^{e-1}$$

in $F(e,n)$, which is the element $x_1^{ne-1} \cdots x_{n-1}^{2e-1} x_n^{e-1}$ that we want. QED

For any symmetric polynomial $f \in \mathbf{Z}[x_1, \ldots, x_n]$, we can view f as an element of the ring $\mathbf{Z}[e_1, \ldots, e_n]$ of symmetric functions and hence of its quotient ring $C(e, n)$. Suppose that f is homogeneous of dimension $\binom{n+1}{2}(e-1)$, the top dimension of the ring $C(e, n)$. Let us define the *degree* of f to be the unique integer $\deg(f)$ such that

$$f = \deg(f)(e_1 \ldots e_n)^{e-1}$$

in $C(e, n)$. Recall the definition

$$\Delta := \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) x_1^{\sigma(n)-1} \cdots x_n^{\sigma(1)-1},$$

which we now view as defining a polynomial in $x_1, \ldots, x_n$.

Lemma 1.3 *Let $f \in \mathbf{Z}[x_1, \ldots x_n]$ be a symmetric polynomial which is homogeneous of degree $\binom{n+1}{2}(e-1)$. Then the integer $\deg(f)$ defined above is given by the formula*

$$\deg(f) = (f\Delta)_{ne-1, \ldots, 2e-1, e-1},$$

where the subscript denotes the coefficient of a given monomial in a polynomial in $x_1, \ldots, x_n$.

Proof. Since the polynomial f is symmetric and the polynomial Δ is antisymmetric, the polynomial $f\Delta$ is antisymmetric. In particular, for each σ in S_n, the coefficient of the monomial $x_1^{\sigma(n)e-1} \cdots x_n^{\sigma(1)e-1}$ in $f\Delta$ is equal to $\operatorname{sgn}(\sigma)$ times the coefficient of $x_1^{ne-1} \cdots x_n^{e-1}$, which is the coefficient considered in the lemma. Call that coefficient $c(f)$.

We now view $f\Delta$ as an element of the quotient ring $F(e, n)$ of $\mathbf{Z}[x_1, \ldots, x_n]$. By Lemma 1.2, the only coefficients of $f\Delta$ which make a difference here are those discussed in the previous paragraph. Precisely, it follows from that lemma and the previous paragraph that

$$f\Delta = n!\, c(f) x_1^{ne-1} \cdots x_n^{e-1}$$

in the ring $F(e, n)$. In particular, this formula shows that $c(f)$ only depends on the image of f in the ring $C(e, n)$ inside $F(e, n)$. We defined $\deg(f)$ to be the integer such that $f = \deg(f)(e_1 \cdots e_n)^{e-1}$ in $C(e, n)$.

Thus, to show that $\deg(f) = c(f)$, it suffices to show that $c(f) = 1$ for $f = (e_1 \cdots e_n)^{e-1}$. Here we view f as a symmetric polynomial in $x_1, \ldots, x_n$, where the e_i's are the elementary symmetric polynomials. By definition, $c(f)$ is the coefficient of $x_1^{ne-1} \cdots x_n^{e-1}$ in the polynomial $f\Delta$. The only monomial in $f\Delta$ that contributes to this coefficient is

$$(x_1)^{e-1}(x_1 x_2)^{e-1} \cdots (x_1 \cdots x_n)^{e-1}(x_1^{n-1} \cdots x_{n-1}),$$

where the last factor comes from Δ. It follows that $c(f) = 1$. QED

Now we can prove Theorem 0.1. By Lemma 1.3, the degree of the ring $C(e, n)$, or equivalently the degree of the element $e_1^{\binom{n+1}{2}(e-1)}$ in $C(e, n)$, is equal to

$$(e_1^{\binom{n+1}{2}(e-1)} \Delta)_{ne-1, \ldots, 2e-1, e-1}.$$

This is exactly what Frobenius's character formula gives as the dimension of a certain irreducible representation of a symmetric group, namely the representation of $S_{\binom{n+1}{2}(e-1)}$ which corresponds to the partition

$$(n(e-1), \ldots, 2(e-1), e-1).$$

In general, Frobenius's formula says that for partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ and ρ of a number N, the character χ^λ is given on an element of S_N of cycle type ρ is equal to the coefficient of $x^{\lambda+\delta}$ in $p_\rho \Delta$, where

$$\begin{aligned} p_\rho &= p_{\rho_1} p_{\rho_2} \cdots \\ p_k &= \sum_{i=1}^{n} x_i^k \\ \delta &= (n-1, n-2, \dots, 0) \\ \Delta &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\sigma(x^\delta). \end{aligned}$$

See Fulton and Harris [6], p. 49 or Macdonald [10], eq. I.7.8.

We can then, following Frobenius (see [6], pp. 49-50 or [10], example I.7.6), evaluate the above coefficient to get a more explicit formula for the dimension of the irreducible representation of S_N corresponding to a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ of N: let $\mu = \lambda + \delta$ (so $\mu_i = \lambda_i + n - i$, $1 \leq i \leq n$), and then the dimension of the representation χ^λ of S_N is

$$\frac{N!}{\prod \mu_i!} \prod_{i<j} (\mu_i - \mu_j).$$

For $\lambda = (n(e-1), (n-1)(e-1), \dots, e-1)$, this equals

$$e^{\binom{n}{2}} \frac{(\binom{n+1}{2}(e-1))!\, 1!\, 2! \cdots (n-1)!}{(e-1)!\,(2e-1)! \cdots (ne-1)!},$$

as we want. QED (Theorem 0.1)

2 The Hall-Littlewood basis for the ring $C(e,n)$ and the related ring $B(e,n)$

In this section we describe a basis for the ring $C(e,n)$ given by Hall-Littlewood functions. As an application, we get a different and more combinatorial formula for the degree of $C(e,n)$ in the sense of Theorem 0.1, which in particular shows that this degree is a multiple of $e^{\binom{n}{2}}$ for e prime. The interest of this fact is that, for $e = 2$, it is explained by the formula

$$\deg Sp(2n)/U(n) = 2^{\binom{n}{2}} \deg SO(2n+1)/U(n),$$

where each variety is considered in its basic projective embedding. We can offer a similar explanation here for any prime number e, using a ring $B(e,n)$ which in the case $e = 2$ is the integral cohomology of $SO(2n+1)/U(n)$.

Let e be any integer at least 2. At first we will take $n = \infty$, which means that we will study the ring

$$C(e) := \mathbf{Z}[e_1, e_2, \dots]/(e_i(x_1^e, x_2^e, \dots), i \geq 1).$$

Macdonald's book [10], chapter III, defines two types of Hall-Littlewood functions, $P_\lambda(x,t)$ and $Q_\lambda(x,t)$, which are symmetric functions in $x = (x_1, x_2, \dots)$ depending on a parameter t. Specializing t to a prime power q, these functions are closely related to the characters of the finite general linear group $GL_n\mathbf{F}_q$, while for $t = -1$ we get Schur's Q-functions which describe the characters of the projective representations of the symmetric group, as well as the Schubert basis for the cohomology of the isotropic Grassmannian. To study the ring $C(e)$, we will specialize t to be a primitive eth root of unity.

For completeness, we recall the definition of the Hall-Littlewood functions. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of length $\leq n$ (so some of the λ_i may be zero), let $m_i(\lambda)$ be the number of λ_j equal to i, for $i \geq 0$. Define

$$v_m(t) = \prod_{i=1}^{m} \frac{1-t^i}{1-t}$$

and

$$v_\lambda(t) = \prod_{i \geq 0} v_{m_i(\lambda)}(t).$$

Then the Hall-Littlewood symmetric function $P_\lambda(x,t)$ is defined by

$$P_\lambda(x_1, \ldots, x_n, t) = \frac{1}{v_\lambda(t)} \sum_{w \in S_n} w(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i<j} \frac{x_i - tx_j}{x_i - x_j}).$$

Here $P_\lambda(x,t)$ is a polynomial with integer coefficients in $x_1, \ldots, x_n, t$, symmetric in $x_1, \ldots, x_n$. We can let the number of variables n go to infinity, to define elements $P_\lambda(x,t) \in \Lambda[t]$, where $\Lambda = \mathbf{Z}[e_1, e_2, \ldots]$ is the ring of symmetric functions. Finally, the functions Q_λ, also called Hall-Littlewood symmetric functions, are defined by

$$Q_\lambda(x,t) = b_\lambda(t) P_\lambda(x,t),$$

where

$$b_\lambda(t) = \prod_{i \geq 1} \varphi_{m_i(\lambda)}(t)$$

and

$$\varphi_m(t) = \prod_{i=1}^{m} (1-t^i).$$

We will need the Pieri formula for multiplying the Hall-Littlewood function $Q_\mu(x,t)$ by $q_r(x,t) := Q_{(r)}(x,t)$ ([10], eq. III.5.7', [11]):

Lemma 2.1

$$Q_\mu q_r = \sum_\lambda \psi_{\lambda/\mu}(t) Q_\lambda,$$

summed over all partitions λ such that $\lambda - \mu$ is a horizontal r-strip, meaning a set of r boxes with no two in the same column. Here

$$\psi_{\lambda/\mu}(t) = \prod_{j \in J} (1 - t^{m_j(\mu)}),$$

where J is the set of integers $j \geq 1$ such that $\lambda - \mu$ has a box in column $j+1$ but not in column j.

Let us now specialize t to be a primitive eth root of unity ζ, for any $e \geq 2$, as was first suggested by Morris [12]. The functions $P_\lambda(x)$ form a basis for the ring Λ of symmetric functions for any value of t, but this is not so for the functions Q_λ (which are multiples of P_λ). In particular, for $t = \zeta$, $Q_\lambda = 0$ unless λ is e-regular, meaning that no part of λ occurs e or more times. The functions Q_λ with λ e-regular form a free $\mathbf{Z}[\zeta]$-basis for a $\mathbf{Z}[\zeta]$-submodule $C'(e)$ of $\Lambda \otimes_{\mathbf{Z}} \mathbf{Z}[\zeta]$. In fact, $C'(e)$ is the $\mathbf{Z}[\zeta]$-subalgebra of $\Lambda \otimes_{\mathbf{Z}} \mathbf{Z}[\zeta]$ generated by the elements q_i, as one can deduce from Lemma 2.1. We can describe $C'(e) \otimes_{\mathbf{Z}[\zeta]} \mathbf{Q}(\zeta)$ as the polynomial subring of $\Lambda \otimes_{\mathbf{Z}} \mathbf{Q}(\zeta)$ generated by the power sums p_i such that $i \not\equiv 0 \pmod{e}$, by Macdonald's example III.7.7. Integrally, the ring $C'(e)$ is more complex. In fact, it is essentially the ring

$$C(e) = \mathbf{Z}[e_1, e_2, \ldots]/(e_i(x_1^e, x_2^e, \ldots), i \geq 1)$$

we want to study. To be precise:

Lemma 2.2 *There is an isomorphism*

$$C(e) \otimes_{\mathbf{Z}} \mathbf{Z}[\zeta] \to C'(e)$$

which sends e_i *in* $C(e)$ *to* q_i *in* $C'(e)$.

Proof. We know that the algebra $C'(e)$ is generated by the elements q_i. So mapping e_i to q_i gives a surjection

$$\varphi : \mathbf{Z}[\zeta][e_1, e_2, \dots] \to C'(e).$$

We need to show that the elements $e_i(x_1^e, x_2^e, \dots)$ for $i \geq 1$ map to 0. Clearly φ maps the power series $E(u) = \sum e_i u^i$ to $Q(u) = \sum q_i u^i$. We also write $H(u) = \sum h_i u^i$ where h_i is the ith complete symmetric function, as in Macdonald [10]. We have $Q(u) = H(u)/H(\zeta u)$ by [10], eq. III.2.10. It follows that

$$\prod_{j=0}^{e-1} Q(\zeta^j u) = 1.$$

Therefore $\prod_{j=0}^{e-1} E(\zeta^j u)$ maps to 1 under the homomorphism φ. We can evaluate this product. We have $E(u) = \prod_{i \geq 1}(1 + x_i u)$, and so

$$\begin{aligned}\prod_{j=0}^{e-1} E(\zeta^j u) &= \prod_{i \geq 1}(1 + x_i u)(1 + \zeta x_i u) \cdots (1 + \zeta^{e-1} x_i u)\\ &= \prod_{i \geq 1}(1 + (-1)^{e-1} x_i^e u^e)\\ &= \sum_{k \geq 0}(-1)^{k(e-1)} e_k(x_1^e, x_2^e, \dots) u^{ke}.\end{aligned}$$

Thus φ sends $e_k(x_1^e, x_2^e, \dots)$ to 0 for $k \geq 1$, so φ is a homomorphism $C(e) \otimes_{\mathbf{Z}} \mathbf{Z}[\zeta] \to C'(e)$. We have already checked that φ is surjective, and since the two rings are both torsion-free and have the same Hilbert series, φ is an isomorphism. QED

It follows that if we think of $C(e, n)$ as the quotient ring of $C(e)$ defined by setting $e_i = 0$ for $i > n$, then $C(e, n) \otimes_{\mathbf{Z}} \mathbf{Z}[\zeta]$ is isomorphic to the quotient ring of $C'(e)$ defined by setting $q_i = 0$ for $i > n$. The latter ring is free over $\mathbf{Z}[\zeta]$ with basis given by the Q_λ's with λ e-regular and $\lambda_1 \leq n$, as follows easily from Lemma 2.1. So those Q_λ's provide a basis for the ring $C(e, n) \otimes_{\mathbf{Z}} \mathbf{Z}[\zeta]$ of interest in this paper.

For $e = 2$, the functions $Q_\lambda(x) = Q_\lambda(x, -1)$ are Schur's Q-functions, which provide a basis for $C(2, n) = H^*(Sp(2n)/U(n), \mathbf{Z})$. This is exactly the basis given by Schubert cells, by Józefiak [9] or Pragacz [13]. As such, it has several good properties. For example, the product of any two basis elements is a nonnegative linear combination of basis elements; there are explicit combinatorial formulas for these coefficients, analogous to the Pieri and Littlewood-Richardson formulas ([8], [10]); and the basis is self-dual.

For $e > 2$, the Hall-Littlewood basis for $C(e, n) \otimes \mathbf{Z}[\zeta]$ has some but not all of these properties. There are again explicit formulas for the product of two elements, generalizing the Pieri and Littlewood-Richardson formulas for $e = 2$, but now the coefficients lie in $\mathbf{Z}[\zeta]$, so it no longer makes sense to ask if they are nonnegative. Also, the basis is not self-dual: that is, a basis element can have a nonzero product with more than one basis element in the complementary dimension.

Nonetheless, it is interesting to see the formula for the degree of the ring $C(e, n)$, as computed in Theorem 0.1, which we find using the Hall-Littlewood basis. The most informative version of this formula is obtained by relating $C(e, n)$ to a slightly different ring. Let $B(e)$ be the $\mathbf{Z}[\zeta]$-submodule of $\Lambda \otimes_{\mathbf{Z}} \mathbf{Z}[\zeta]$ spanned by the Hall-Littlewood functions P_λ for e-regular partitions λ; it is an algebra by [10], example III.7.7. Let $B(e,n)$ be the quotient of the ring $B(e)$ by the elements P_λ with $\lambda_1 > n$. It is clear that

$$C(e,n) \otimes_{\mathbf{Z}} \mathbf{Q}(\zeta) = B(e,n) \otimes_{\mathbf{Z}[\zeta]} \mathbf{Q}(\zeta).$$

For $e = 2$, $B(e,n)$ is the integral cohomology ring of the spinor variety $SO(2n+1)/U(n)$, and in fact the basis P_λ corresponds to the Schubert basis for this ring [9], [13]. We define the degree of the ring $C(e,n)$, as in Theorem 0.1, to be the coefficient of $e_1^{\binom{n+1}{2}(e-1)}$ as a multiple of $(e_1 \cdots e_n)^{e-1}$. Likewise, define the degree of $B(e,n)$ to be the coefficient in $\mathbf{Z}[\zeta]$ of $P_1^{\binom{n+1}{2}(e-1)}$ as a multiple of P_λ, where $\lambda = (1^{e-1}2^{e-1}\cdots n^{e-1})$.

Recall that a standard tableau of shape λ, for a partition λ, is a numbering of the boxes in the diagram of λ by the numbers from 1 to $|\lambda|$ which is increasing in both rows and columns.

Lemma 2.3

$$\deg C(e,n) = (1-\zeta)^{\binom{n+1}{2}(e-1)} e^{-n} \deg B(e,n),$$

and

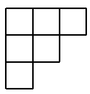
$$\deg B(e,n) = \sum_T \alpha_T(\zeta).$$

The sum runs over the standard tableaux of shape $\lambda = (1^{e-1}2^{e-1}\cdots n^{e-1})$, *and*

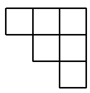
$$\alpha_T(\zeta) := \prod_{1\le i\le N} \frac{1-\zeta^{m_{j_i}(\lambda^{(i)})}}{1-\zeta}.$$

Here $N = \binom{n+1}{2}(e-1)$, $\lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(N)} = \lambda$ *is the sequence of partitions corresponding to the tableau* T, *and* $j_i \ge 0$ *is the integer such that the box* $\lambda^{(i)} - \lambda^{(i-1)}$ *is in column* j_i. *Notice that* $\alpha_T(\zeta) = 0$ *if any of the partitions* $\lambda^{(i)}$ *is not* e*-regular, so it suffices to sum over the* e*-regular standard tableaux* T.

For $e = 2$, so that $\zeta = -1$, Lemma 2.3 gives the interpretation of the degree of the isotropic Grassmannian mentioned in the introduction. Indeed, $\alpha_T(-1) = 1$ for all 2-regular tableaux T of shape $(n, n-1, \ldots, 1)$, and so the lemma says that the degree of $C(2,n)$ is $2^{\binom{n}{2}}$ times the number of 2-regular tableaux of shape $(n, n-1, \ldots, 1)$, or equivalently times the number of shifted standard tableaux of that shape. Here the usual diagram of the partition $(n, n-1, \ldots, 1)$ is



and the shifted diagram is:



A shifted standard tableau is a numbering of the shifted diagram by the numbers from 1 to $\binom{n+1}{2}$ which is increasing in rows and columns. The resulting formula

$$\frac{\binom{n+1}{2}!\, 1!\, 2! \cdots (n-1)!}{1!\, 3! \cdots (2n-1)!}$$

for the number of shifted standard tableaux of shape $(n, n-1, \ldots, 1)$ goes back to Schur [17]; see also Macdonald [10], example III.8.12.

For $e \ge 3$, Lemma 2.3 is a less satisfactory description of the degree of $C(e,n)$. At least it implies the following statement.

Corollary 2.4 *For any prime number e, the degree of $C(e,n)$ is a multiple of $e^{\binom{n}{2}}$.*

Proof. Since e is a prime number, $(1-\zeta)^{e-1}$ is equal to e times a unit in $\mathbf{Z}[\zeta]$. So $(1-\zeta)^{\binom{n+1}{2}(e-1)}e^{-n}$ is equal to $e^{\binom{n}{2}}$ times a unit in $\mathbf{Z}[\zeta]$. So Lemma 2.3 shows that the integer $\deg C(e,n)$ is a multiple of $e^{\binom{n}{2}}$. QED

Proof of Lemma 2.3. We think of $C(e,n)\otimes_{\mathbf{Z}}\mathbf{Z}[\zeta]$ as a quotient of the algebra $C(e)\otimes\mathbf{Z}[\zeta]$ spanned by Hall-Littlewood functions Q_μ. In these terms, the element e_1 of $C(e,n)$ is equal to q_1, and the Pieri formula, Lemma 2.1, implies that $(e_1\cdots e_n)^{e-1}$ is equal to Q_λ, where $\lambda=(1^{e-1}2^{e-1}\cdots n^{e-1})$. So the degree of $C(e,n)$ is equal to the coefficient of Q_λ in the expansion of q_1^N as a linear combination of Hall-Littlewood functions Q_μ. Since $q_1=(1-\zeta)P_1$ and

$$\begin{aligned}Q_\lambda &= [(1-\zeta)\cdots(1-\zeta^{e-1})]^n P_\lambda\\ &= e^n P_\lambda,\end{aligned}$$

we have

$$\deg C(e,n)=(1-\zeta)^{\binom{n+1}{2}(e-1)}e^{-n}\deg B(e,n).$$

To compute $\deg B(e,n)$, it is convenient to use the version of the Pieri formula for the Hall-Littlewood functions P_μ, from [10], eq. III.5.7:

$$P_\mu(x,t)P_1(x,t)=\sum_\lambda \alpha_{\lambda/\mu}(t)P_\lambda,$$

summed over all partitions λ such that $\lambda-\mu$ is a box in some column j, with $\alpha_{\lambda/\mu}(t):=(1-t^{m_j(\lambda)})/(1-t)$. Taking $t=\zeta$, this implies the formula for $\deg B(e,n)$ as a sum over standard tableaux. QED

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