Configuration spaces of algebraic varieties

Burt Totaro

This paper determines the rational cohomology ring of the configuration space of n-tuples of distinct points in a smooth complex projective variety X. The answer depends only on the cohomology ring of X.

1 Introduction

Let X be a topological space. We will consider the configuration space of ordered n-tuples of points in X:

$$F(X,n) = \{(x_1,\ldots,x_n) \in X^n : x_i \neq x_j \text{ if } i \neq j\}.$$

The symmetric group S_n acts freely on $F(X,n)$; let $B(X,n) = F(X,n)/S_n$ be the space of unordered n-tuples of distinct points in X.

The most famous of these spaces is $B(\mathbf{R}^2,n)$: Fadell and Neuwirth showed that this space is an Eilenberg-Mac Lane space $K(\pi,1)$, with fundamental group isomorphic to Artin's braid group on n strings [12]. So the cohomology of $B(\mathbf{R}^2,n)$ is just the cohomology of the braid group; it was computed by Fuks [13] and Cohen [7]. The space $F(\mathbf{R}^2,n)$ is also a $K(\pi,1)$ space, with fundamental group equal to the "colored braid group," that is, the kernel of the natural map from the braid group onto the symmetric group. Arnold computed the cohomology of $F(\mathbf{R}^2,n)$; clearly this space is just the complement of a finite union of complex hyperplanes in $\mathbf{C}^n$ [1].

The space $B(\mathbf{R}^m,n)$ is not a $K(\pi,1)$ for $m>2$, but it does have an interesting homotopy type: as n goes to infinity, the cohomology of $B(\mathbf{R}^m,n)$ approximates the cohomology of the infinite-dimensional space $\Omega^m S^m$, the space of maps from the m-sphere to itself. This fact has many consequences in homotopy theory; see F. Cohen [7] and R. Cohen [10] for surveys.

Recently Bödigheimer, Cohen, and Taylor [6] have computed the homology groups of $B(X,n)$, for any odd-dimensional manifold X. Also, Löffler and Milgram [18] computed the $\mathbf{Z}/2$-homology groups of $B(X,n)$ for any manifold X. The answers depend only on the homology of X (along with the dimension of X). By contrast, the rational Betti numbers of $B(X,n)$ are not determined by the rational Betti numbers of X for even-dimensional manifolds X, by the example in section 5. The situation is worse for cohomology rings; we do not even know whether the cohomology ring of $B(X,n)$ is a homotopy invariant of closed manifolds X. On the other hand, Bendersky and Gitler observed that the cohomology *groups* of $F(X,n)$ and $B(X,n)$ are homotopy invariants of closed oriented manifolds X, and for rationally formal manifolds X, such as smooth complex projective varieties, they computed the rational cohomology groups of $F(X,n)$ and $B(X,n)$ explicitly in terms of the rational cohomology ring of X [3].

In this paper we find that if X is a smooth complex projective variety of complex dimension l, then the rational cohomology rings of $F(X,n)$ and $B(X,n)$ can be computed from the rational cohomology ring of X. This improves Fulton-MacPherson's theorem [14] that these cohomology rings can be computed from the rational cohomology ring of X *and* the Chern classes of X: the Chern classes are actually irrelevant. This result was found at the same time by Kriz [17], who proved it by algebraically simplifying Fulton-MacPherson's description of the cohomology ring.

Fulton and MacPherson's computation was an application of their compactification of $F(X,n)$, $F(X,n) \hookrightarrow X[n]$. There is an obvious compactification $F(X,n) \hookrightarrow X^n$, but in algebraic geometry one usually prefers to compactify a noncompact variety so that it becomes the complement of a divisor with normal crossings, and this is what Fulton and MacPherson did. When X is the projective line, their compactification is related to the moduli space $\overline{M}_{0,n}$ of "n-pointed stable curves of genus zero," defined by Grothendieck in SGA 7 and studied by Knudsen [16].

The point of this paper is that the rational cohomology of $F(X,n)$ can be computed just from the naive compactification $F(X,n) \hookrightarrow X^n$.

The rational cohomology rings of $F(X,n)$ and $B(X,n)$ are described explicitly in section 5, based on the work in earlier sections. The method is to consider the Leray spectral sequence for the open inclusion $F(X,n) \hookrightarrow X^n$. This spectral sequence, which converges to $H^*F(X,n)$, makes sense for any space X. Its E_2 term and first nontrivial differential can be explicitly described for any oriented real manifold X. This spectral sequence was described by Cohen and Taylor ([8], pp. 117-118), although they did not mention that it was simply a Leray spectral sequence. The E_2 term has a very pleasant structure, which we describe in detail after the statement of Theorem 1.

The surprise is that this spectral sequence degenerates after the first nontrivial differential, provided that we use rational coefficients and that X is a smooth complex projective variety. A similar phenomenon is well known for inclusions $U \hookrightarrow Y$ when $Y - U$ is a divisor with normal crossings in a smooth projective variety Y; but for $\dim_{\mathbf{C}} X > 1$, the subset $X^n - F(X,n)$ is not a divisor, and its singularities are not just normal crossings.

This paper was inspired by the work of Cohen, Taylor, Fulton, and MacPherson. J. D. Stasheff and the referee made useful suggestions for improving the exposition.

2 The Leray spectral sequence

For $a \neq b \in \{1, \ldots, n\}$, let $p_a : H^*(X) \to H^*(X^n)$ and $p_{ab}^* : H^*(X^2) \to H^*(X^n)$ be the obvious pullbacks. For X an oriented real manifold of dimension m, let $\Delta \in H^m(X^2)$ denote the class of the diagonal.

For the following theorem we need some combinatorial definitions. Define a partition J of a set $\{1, \ldots, n\}$ to be a set of nonempty subsets of $\{1, \ldots, n\}$ which are pairwise disjoint and whose union is $\{1, \ldots, n\}$. In particular, we do not specify an order on the set J. For any space X, a partition J of $\{1, \ldots, n\}$, say into $n-r$ elements, determines a "diagonal" subspace $X_J \subset X^n$, defined as

$$X_J = \{(x_1, \ldots, x_n) \in X^n : x_i = x_j \text{ if } i \text{ and } j \text{ belong to the same element of } J\}.$$

We will often write X_J^{n-r} for this subspace, although we do not specify an identification of this diagonal subspace with X^{n-r}; that would depend on an order of the set J. Finally, for a partition J of $\{1, \dots, n\}$ into subsets of orders $j_1, \dots, j_{n-r}$, define

$$c_J = (j_1 - 1)! \cdots (j_{n-r} - 1)!$$

Theorem 1 *Let X be an oriented real manifold of dimension m. Let k be a field. The inclusion $F(X,n) \hookrightarrow X^n$ determines a Leray spectral sequence which converges to $H^*(F(X,n);k)$ as an algebra. The E_2 term is a bigraded algebra which we now describe. It is the quotient of the graded-commutative k-algebra*

$$H^*(X^n;k)[G_{ab}],$$

where $H^i(X^n)$ has degree $(i,0)$ and the G_{ab} are generators of degree $(0, m-1)$ for $1 \le a, b \le n$, $a \ne b$, modulo the following relations:

$$\begin{array}{lrcl} (0) & G_{ab} & = & (-1)^m G_{ba} \\ (1) & (G_{ab})^2 & = & 0 \\ (2) & G_{ab}G_{ac} + G_{bc}G_{ba} + G_{ca}G_{cb} & = & 0 \text{ for } a,b,c \text{ distinct} \\ (3) & p_a^*(x)G_{ab} & = & p_b^*(x)G_{ab} \text{ for } a \ne b, x \in H^*X. \end{array}$$

The differential is given by

$$dG_{ab} = p_{ab}^* \Delta.$$

The symmetric group S_n acts on $F(X,n)$ and X^n, and so it acts on this Leray spectral sequence. The action on the E_2 term is given by the obvious action on $H^*(X^n)$ and by $\sigma(G_{ab}) = G_{\sigma(a)\sigma(b)}$.

In fact, we can describe this E_2 term with $\mathbf{Z}$ coefficients as well. The only nonzero rows are the $r(m-1)$st rows, $0 \le r \le n-1$. The $r(m-1)$st row is the direct sum, over all partitions J of $\{1, \dots, n\}$ into $n-r$ pieces, of the cohomology of X_J^{n-r} with integer coefficients, tensored with $\mathbf{Z}^{c_J}$.

$$\begin{array}{ccccc} \vdots & & & & \\ 2(m-1) & \oplus_J H^0(X_J^{n-2};\mathbf{Z}) \otimes \mathbf{Z}^{c_J} & \oplus_J H^1(X_J^{n-2};\mathbf{Z}) \otimes \mathbf{Z}^{c_J} & \oplus_J H^2(X_J^{n-2};\mathbf{Z}) \otimes \mathbf{Z}^{c_J} & \cdots \\ m-1 & \oplus_J H^0(X_J^{n-1};\mathbf{Z}) & \oplus_J H^1(X_J^{n-1};\mathbf{Z}) & \oplus_J H^2(X_J^{n-1};\mathbf{Z}) & \cdots \\ 0 & H^0(X^n;\mathbf{Z}) & H^1(X^n;\mathbf{Z}) & H^2(X^n;\mathbf{Z}) & \cdots \end{array}$$

This spectral sequence converges to $H^*(F(X,n);\mathbf{Z})$.

In more detail, the only nonzero rows are the $r(m-1)$st rows, $0 \le r \le n-1$. The $r(m-1)$st row is a sum over all partitions J of the set $\{1, \dots, n\}$ into $n-r$ pieces; clearly each such partition corresponds to a diagonal $X_J^{n-r} \subset X^n$. For a partition J into subsets of orders $j_1, \dots, j_{n-r}$, $\sum j_k = n$, the number c_J which occurs in the above E_2 term is

$$c_J = (j_1 - 1)! \cdots (j_{n-r} - 1)!$$

Remarks. (1) As we mentioned in the introduction, this spectral sequence was first described by Cohen and Taylor [8]. Also, they observed that the spectral sequence has no differentials (with integer coefficients) if X is the product of any manifold with the real line, and that the integer cohomology ring of $F(X,n)$ is isomorphic to the E_2 term as a ring (not just modulo filtration) if X is the product of any manifold with $\mathbf{R}^2$.

(2) The E_2 term of this spectral sequence has a rich and somehow very natural structure, as I will explain.

There are two types of generators in cohomology, one type, the classes G_{ab}, coming from the cohomology of the configuration space of n-tuples of points in Euclidean space, because X locally looks like Euclidean space, and the other type coming from the cohomology of X. The relations come in two types, relations (0)-(2) above coming from the locally Euclidean structure of X, and relation (3) which connects the classes G_{ab} with the classes coming from the cohomology of X.

The relations giving the vanishing of the squares of the classes G_{ab} and the three-term relation for the G_{ab}'s are related to the antisymmetry law and the Jacobi identity in graded Lie algebras, as F. Cohen discovered ([7], p. 188). The phrase "graded Lie algebra" which occurs in his statement has to be explained more precisely, however, as we will now do. To state the result, we use that the cohomology ring of $F(\mathbf{R}^m, n)$ is generated by the G_{ab}'s modulo relations (0)-(2). Let $T(n, m)$ be the integer cohomology group of $F(\mathbf{R}^m, n)$ in the top dimension $(= (m-1)(n-1))$, with its natural action of the symmetric group S_n. Then for a module V over a commutative ring k, the nth graded piece of the free non-associative algebra over V with product $[x, y]$ satisfying $[x, y] = -[y, x]$ and the Jacobi identity is given by

$$T(n, m)^* \otimes_{S_n} V^{\otimes n}$$

for any odd m. (If k is a field of characteristic $\neq 2$, this is the nth graded piece of the free Lie algebra on V, but in characteristic 2 we have omitted the relation $[x, x] = 0$.) Likewise, for m even, the representations $T(n, m)^*$ of S_n define in the same way the free graded non-associative algebra on V in degree 1 which satisfies $[x, y] = -(-1)^{|x||y|}[y, x]$ and the graded Jacobi identity. This is the free graded Lie algebra in characteristics $\neq 2$, 3, but the good definition of graded Lie algebras in characteristics 2 and 3 is more complicated [2]. The proof goes along the lines suggested by Cohen, but it can be simplified so that it uses only the cohomology of the spaces $F(\mathbf{R}^m, n)$, rather than the homology operations on iterated loop spaces. Also, these results are mentioned in Cohen-Taylor [9], p. 95, with $T(n, m)^*$ replaced by $T(n, m)$, which is just a misquotation of Cohen's theorem.

Proof of Theorem 1. The Leray spectral sequence for the inclusion $f : F(X, n) \hookrightarrow X^n$ has the form

$$H^i(X^n; R^j f_* \mathbf{Z}) \Rightarrow H^{i+j}(F(X, n); \mathbf{Z}).$$

Here $R^j f_* \mathbf{Z}$ is the sheaf on X^n associated to the presheaf

$$U \mapsto H^j(U \cap F(X, n); \mathbf{Z}),$$

where U runs over the open subsets of X^n. We need to see what these sheaves are.

The stalk of $R^j f_* \mathbf{Z}$ at a point $x \in X^n$ is easy to describe. Suppose that $x = (x_1, \ldots x_1, \ldots, x_s, \ldots, x_s)$, where $x_1, \ldots, x_s \in X$ are distinct and each point x_j occurs i_j times, so that $\sum i_j = n$. Then

$$(R^j f_* \mathbf{Z})_x = H^j(U \cap F(X, n); \mathbf{Z}),$$

where U is a nice small neighborhood of $x \in X^n$, and this is

$$\cong H^j(F(T_{x_1}X, i_1) \times \cdots \times F(T_{x_s}X, i_s); \mathbf{Z}).$$

This is a basic observation. It can be seen, for example, by using the exponential map of some riemannian metric on X to identify a small open ball in the tangent space $T_{x_j}X$ with a small neighborhood of x_j in X, small enough to be disjoint from the similar neighborhoods of the other points x_j.

Thus, to describe the E_2 term of the Leray spectral sequence for $F(X, n)$, we need to understand the cohomology of $F(\mathbf{R}^m, n)$. We will quote Cohen's results from [7].

Fix integers m and n with $m > 0$. Define graded-commutative algebras over $\mathbf{Z}$, $G(m, n)$, with generators

$$G_{ab},\ 1 \leq a, b \leq n,\ a \neq b;$$

the degree of G_{ab} is $m-1$. The relations are

$$\begin{array}{lrcl} (0) & G_{ab} & = & (-1)^m G_{ba} \\ (1) & (G_{ab})^2 & = & 0 \\ (2) & G_{ab}G_{ac} + G_{bc}G_{ba} + G_{ca}G_{cb} & = & 0 \text{ for } a, b, c \text{ distinct} \end{array}$$

Thus, if m is even, $G(m, n)$ is a quotient of the exterior algebra on the G_{ab}. Further, define $\sigma G_{ab} = G_{\sigma a, \sigma b}$ for σ in the symmetric group S_n.

Lemma 1 *The integral cohomology of $F(\mathbf{R}^m, n)$ as an algebra over S_n is isomorphic to $G(m, n)$.*

Lemma 2 *A basis for $H^{r(m-1)}F(\mathbf{R}^m, n)$ is given by $G_{a_1b_1}G_{a_2b_2} \cdots G_{a_rb_r}$, where $a_1 < \cdots < a_r$ and $a_i > b_i$ for $i = 1, \ldots, r$. The Poincaré series for $F(\mathbf{R}^m, n)$ is $\prod_{j=1}^{k-1}(1 + jt^{m-1})$.*

In particular, we see that the cohomology of a product $F(\mathbf{R}^m, i_1) \times \cdots \times F(\mathbf{R}^m, i_s)$ is 0 except in dimensions divisible by $m-1$. This implies that the cohomology sheaves $R^jf_*\mathbf{Z}$ for the inclusion $F(X, n) \hookrightarrow X^n$ are 0 except for $j = r(m-1)$, $0 \leq r \leq n-1$. (So the Leray spectral sequence will be zero except in rows $r(m-1)$, $0 \leq r \leq n-1$.) Also, we can see that the sheaf $R^{r(m-1)}f_*\mathbf{Z}$ is supported on the union of the diagonals $X^{n-r} \subset X^n$. We want to express this sheaf as the direct sum of sheaves supported on each of the diagonals $X^{n-r} \subset X^n$ (they are in one-to-one correspondence with partitions of the set $\{1, \ldots, n\}$ into $n-r$ subsets), and luckily this is possible.

For this purpose, we need to analyze the cohomology of $F(\mathbf{R}^m, i_1) \times \cdots \times F(\mathbf{R}^m, i_k)$ in dimension $r(m-1)$. These cohomology groups are nonzero only for $0 \leq r \leq n-k$, and the crucial point for us is that if $r < n-k$, that is, if we are not looking at top-dimensional cohomology, then all $r(m-1)$-dimensional classes are pulled back from similar products $F(\mathbf{R}^m, j_1) \times \cdots \times F(\mathbf{R}^m, j_{n-r})$, on which these classes are top-dimensional. More precisely:

Lemma 3 *Let I be a partition of the set $\{1, \ldots, n\}$, and suppose $r \geq 0$. Then*

$$H^{r(m-1)}(F(\mathbf{R}^m, i_1) \times \cdots \times F(\mathbf{R}^m, i_k); \mathbf{Z}) \cong \oplus_J H^{r(m-1)}(F(\mathbf{R}^m, j_1) \times \cdots \times F(\mathbf{R}^m, j_{n-r}); \mathbf{Z}).$$

More precisely, the natural map from the right side to the left side is an isomorphism. Here the sum is over all partitions J of $\{1,\dots,n\}$ into $n-r$ subsets such that J refines I.

Proof. We will first prove the lemma when I is the trivial partition of $\{1,\dots,n\}$ into just one piece. The general result is an easy consequence, using the Künneth formula.

By Lemma 2, a basis for $H^{r(m-1)}F(\mathbf{R}^m,n)$ is given by monomials $G_{a_1b_1}\cdots G_{a_rb_r}$, $a_1<\cdots<a_r$, $n\geq a_k>b_k\geq 1$ for $k=1,\dots,r$. Such a basis element determines a partition J of $\{1,\dots,n\}$ into exactly $n-r$ sets, by defining $a_k\simeq b_k$ for $k=1,\dots,r$. In fact, it is clear that this equivalence relation partitions $\{1,\dots,n\}$ into at least $n-r$ sets, since we have made r identifications. The condition that the a's are increasing implies that for each $1\leq k\leq r$, we have $a_k>b_k$, and $a_k>a_l$ and $a_k>b_l$ for all $l<k$; so a_k has not been identified to anything before step k (if we think of first identifying a_1 with b_1, then a_2 with b_2, and so on). This implies that the basis element $G_{a_1b_1}\cdots G_{a_rb_r}$ determines a partition J of $\{1,\dots,n\}$ into exactly $n-r$ sets.

If the partition J consists of subsets of orders $j_1,\dots,j_{n-r}$, then it is clear that the basis element $G_{a_1b_1}\cdots G_{a_rb_r}\in H^{r(m-1)}F(\mathbf{R}^m,n)$ lies in the image of the map determined by J, $H^{r(m-1)}F(\mathbf{R}^m,j_1)\times\cdots\times F(\mathbf{R}^m,j_{n-r})\rightarrow H^{r(m-1)}F(\mathbf{R}^m,n)$. Conversely, the image of this map consists of all basis elements $G_{a_1b_1}\cdots G_{a_rb_r}$ such that a_k is in the same class in J as b_k for all $1\leq k\leq r$.

This implies the direct-sum splitting of the theorem in the special case where I is the trivial partition of $\{1,\dots,n\}$. The result follows in general by the Kunneth formula. (The Künneth formula has a simple form here, even with $\mathbf{Z}$ coefficients, because the spaces $F(\mathbf{R}^m,i)$ have torsion-free cohomology.)

QED (Lemma)

Remark. Lemma 3 is a special case of Goresky-MacPherson's splitting theorem for the homology of the complement of a finite union of linear spaces in $\mathbf{R}^n$ ([15], pp. 237-239). They do not work just with the "braid" arrangement as we have, but, more generally, with any arrangement of codimension-k linear subspaces $V_i\subset\mathbf{R}^n$ such that all intersections $V_{i_1}\cap\cdots\cap V_{i_r}$ have codimension a multiple of k. It should be possible to generalize Deligne's explicit description of the rational homotopy type of the complement of a divisor with normal crossings, in order to describe the complement of a subvariety with "Goresky-MacPherson crossings"; this paper is essentially a special case. Would the more general theory have any interesting applications?

We now see what Lemma 3 says about the sheaf $R^{r(m-1)}f_*\mathbf{Z}$ on X^n. For each partition J of the set $\{1,\dots,n\}$ into $n-r$ subsets, of orders $j_1,\dots,j_{n-r}$, consider the inclusion $g_J: F(X,j_1)\times\cdots\times F(X,j_{n-r})\subset X^n$. There is a commutative diagram:

$$\begin{array}{ccc} F(X,n) & \longrightarrow & X^n \\ \downarrow & & \downarrow \\ F(X,j_1)\times\cdots\times F(X,j_{n-r}) & \longrightarrow & X^n \end{array}$$

Define a sheaf ϵ_J over X^n by $\epsilon_J = R^{r(m-1)} g_{J*}\mathbf{Z}$. The commutative diagram gives a map of sheaves

$$\epsilon_J \to R^{r(m-1)} f_* \mathbf{Z},$$

which comes from restriction of cohomology classes,

$$H^{r(m-1)}(U \cap (F(X, j_1) \times \cdots \times F(X, j_{n-r}); \mathbf{Z}) \to H^{r(m-1)}(U \cap F(X, n); \mathbf{Z}),$$

where $U \subset X^n$ is an open subset.

The sheaf ϵ_J is 0 outside the diagonal $X_J^{n-r} \subset X^n$ and locally constant on X_J^{n-r}. To see that it is locally constant, let U be a nice small neighborhood of a point $x \in X_J^{n-r}$. We have

$$\epsilon_J(U) = H^{r(m-1)}(F(U_1, j_1) \times \cdots \times F(U_{n-r}, j_{n-r}); \mathbf{Z}),$$

where $U_i \subset X$ is a nice small neighborhood of x_i; we can assume that each U_i is diffeomorphic to $\mathbf{R}^m$. Clearly ϵ_J is a locally constant sheaf over X_J^{n-r}, since it is just the cohomology of the fibers of a fibration over X_J^{n-r}. It is locally isomorphic to $\mathbf{Z}^{c_J}$, where

$$c_J = (j_1 - 1)! \cdots (j_{n-r} - 1)!$$

Indeed, $F(\mathbf{R}^m, j_1)$ has highest nonzero cohomology group equal to $\mathbf{Z}^{(j_1-1)!}$, and the computation of c_J follows. The lemma implies that the sheaf map $\oplus_{|J|=n-r}\epsilon_J \to R^{r(m-1)} f_* \mathbf{Z}$ is an isomorphism on stalks, so it is an isomorphism.

Up to now, the argument has used only that X is a manifold. We now assume that X is oriented. Then we have an isomorphism $\epsilon_J \cong \mathbf{Z}^{c_J}$ over X_J^{n-r}. This follows from Cohen's calculations of $H^*F(\mathbf{R}^m, n)$: all the classes are products of pullbacks of the generator of $H^{m-1}F(\mathbf{R}^m, 2) \cong H^{m-1}F(\mathbf{R}^m \times \mathbf{R}^m - \Delta; \mathbf{Z}) \cong H^{m-1}(\mathbf{R}^m - 0; \mathbf{Z})$, and this depends only on the orientation of $\mathbf{R}^m$.

Therefore, if X is an oriented m-manifold, we have an isomorphism of sheaves over X^n:

$$R^{r(m-1)} f_* \mathbf{Z} \cong \oplus_{|J|=n-r} \mathbf{Z}^{c_J}_{X_J^{n-r}}.$$

This implies Theorem 1.

3 The first nontrivial differential

Let X be a real oriented m-manifold. From the description of the E_2 term of the Leray spectral sequence for $F(X, n) \hookrightarrow X^n$ in theorem 1, we see that most of the rows are 0. It follows that the first differential which can be nonzero is d_m.

Theorem 2 *The differential d_m takes the generator of $H^0(X^{n-1}; \mathbf{Z})$, for each diagonal $X^{n-1} \subset X^n$, to the cohomology class $[X^{n-1}] \in H^m(X^n; \mathbf{Z})$, that is, to the pullback of the class of the diagonal $[\Delta] \in H^m(X \times X)$. This determines d_m on the whole E_2 term.*

Proof. The E_2 term is generated as an algebra by the bottom row, $H^*(X^n; \mathbf{Z})$, together with the group $\oplus_{1 \le i < j \le n} H^0(X_{ij}^{n-1}; \mathbf{Z})$. Here, for each $1 \le i < j \le n$, we define $X_{ij}^{n-1} = \{(x_1, \ldots, x_n) \in X^n : x_i = x_j\}$. The differential d_m is 0 on the bottom row, since it maps each row to a lower row. So it suffices to determine the map

$$d_m : \oplus H^0(X^{n-1}; \mathbf{Z}) \to H^m(X^n; \mathbf{Z}).$$

Pick a diagonal $X_{ij}^{n-1} \subset X^n$ as above. Then the cohomology class $1 \in H^0(X_{ij}^{n-1}; \mathbf{Z})$ is pulled back from the corresponding class in the Leray spectral sequence for the inclusion $X^n - X_{ij}^{n-1} \subset X^n$, and by functoriality of Leray spectral sequences it is enough to determine $d_m(1) \in H^m(X^n; \mathbf{Z})$ for this latter spectral sequence. But we know what the differentials are for the Leray spectral sequence of $Y - Z \subset Y$, where Z is a smooth submanifold with orientable normal bundle in a smooth manifold Y: they are just Gysin maps. The theorem follows.

Remark. Note that for X a real orientable manifold of dimension m, both $[X_{ij}^{n-1}] \in H^m(X^n)$ and $[\Delta] \in H^m(X^2)$ are only determined up to sign.

4 Degeneration

We don't know any example of an m-manifold X where the Leray spectral sequence for $F(X, n) \hookrightarrow X^n$ in rational cohomology has another differential after the first nontrivial differential, d_m. It would be interesting to know if there are such examples. We will prove, however, that if X is a smooth complex projective variety and we use rational coefficients, then the spectral sequence does degenerate after the first nontrivial differential. Moreover, the $E_{2l+1} = E_\infty$ term of the spectral sequence ($l = \dim_{\mathbf{C}} X$) is isomorphic to the cohomology ring of $F(X, n)$ as a $\mathbf{Q}$-algebra, not just to the associated graded ring with respect to some filtration of $H^*(F(X, n); \mathbf{Q})$.

Theorem 3 *If X is a smooth complex projective variety of complex dimension l, then the rational cohomology ring of $F(X, n)$ is isomorphic to the cohomology of the algebra $E_2 \otimes \mathbf{Q}$, where E_2 is the algebra described in theorem 1, with respect to the differential d_{2l} described in theorem 2.*

Proof. By Deligne [11], the rational cohomology of every algebraic variety has a natural filtration called the weight filtration. This filtration is trivial ($H^i(X; \mathbf{Q})$ is pure of weight i) for smooth projective varieties, and in general the weight filtration on $H^i(X; \mathbf{Q})$ expresses how the cohomology of X can be built from the cohomology of smooth projective varieties. Also, for any algebraic map $f : A \to B$, every group in the Leray spectral sequence converging to H^*A has a weight filtration, and all the differentials are strictly compatible with this filtration.

This compatibility of the weight filtration with the Leray spectral sequence is verified most easily from the definition of the weight filtration in l-adic cohomology ([11], pp. 83-84). Here l is a prime number. Given a variety $A/\mathbf{C}$, choose a finitely generated field $K \subset \mathbf{C}$ over which A is defined, and then the Galois group $Gal(K^{alg}/K)$ acts on $H^*(A; \mathbf{Q}_l)$. Choose a prime m and a Frobenius element $F_m \in Gal(K^{alg}/K)$. Then the eigenvalues α of F_m (in a suitable finite extension of $\mathbf{Q}_l$) are in fact algebraic integers, and for each α, there is an integer $w(\alpha)$ such that all complex conjugates of α have absolute value $N(m)^{w(\alpha)/2}$. Let ${}_mW_j$ be the sum of the eigenspaces corresponding to the eigenvalues α of F_m with $w(\alpha) = j$. Then the filtration by the $\oplus_{j \le i}\, {}_mW_j$ is independent of m and of the choice of F_m; it comes from a filtration of $H^*(A; \mathbf{Q})$ which is independent of l; this is the weight filtration.

Since the Galois group acts on all l-adic cohomology groups, maps, and spectral sequences, we deduce immediately that the differentials in the Leray spectral sequence with $\mathbf{Q}$ coefficients are strictly compatible with the weight filtration.

Next, we need to describe explicitly the weight filtration on $H^i(X^n; R^j f_* \mathbf{Q})$, the E_2 term of the Leray spectral sequence. Let $l = \dim_{\mathbf{C}} X$. By Theorem 1, $R^j f_* \mathbf{Q} = 0$ if j is not divisible by $2l-1$, and

$$R^{r(2l-1)} f_* \mathbf{Q} = \oplus_{|J|=n-r} \mathbf{Q}^{c_J}_{X^{n-r}}.$$

Since $X^{n-r} \subset X^n$ is a smooth subvariety of codimension rl, there is a Gysin isomorphism

$$H^i(X^{n-r}; \mathbf{Q})(-rl) \cong H^i(X^n; \mathbf{Q}_{X^{n-r}}).$$

The $(-rl)$ denotes a shift by $-2rl$ for the weight filtration, so that $W(n)_k = W_{k+2n}$. The group $H^i(X^{n-r}; \mathbf{Q})$ is pure of weight i, because X^{n-r} is a smooth projective variety. So $H^i(X^n; \mathbf{Q}_{X^{n-r}})$ and hence $H^i(X^n; R^{r(m-1)} f_* \mathbf{Q})$ are pure of weight $i + 2rl$.

The differential d_j, $j \geq 2$, in the Leray spectral sequence can only be nonzero for $j = k(2l-1)+1$. In this case, it maps a subquotient of $H^i(X^n; R^{r(2l-1)} f_* \mathbf{Q})$ to a subquotient of $H^{i+k(2l-1)+1}(X^n; R^{(r-k)(2l-1)} f_* \mathbf{Q})$. That is, it maps a group which is pure of weight $i+2rl$ to one which is pure of weight $i+k(2l-1)+1+2(r-k)l = i+2rl+(1-k)$. Such a map must be zero unless $k=1$. So the only differential which can be nonzero is d_{2l} and we have proved the degeneration of the spectral sequence.

This implies, by the well-known properties of the Leray spectral sequence, that there is a filtration of the ring $H^*(F(X,n); \mathbf{Q})$ such that the $E_{2l+1} = E_\infty$ term of the spectral sequence is isomorphic to the associated graded ring with respect to this filtration. But the weights of the groups contributing to a given $H^i(F(X,n); \mathbf{Q})$ are all different, and so this filtration is precisely the weight filtration of $H^*(F(X,n); \mathbf{Q})$, except for trivial changes corresponding to the fact that the only nonzero rows in the spectral sequence are those numbered $r(2l-1)$ for some r. Now Deligne proves ([11], p. 81) that the cohomology ring of any algebraic variety is isomorphic to the associated graded ring with respect to the weight filtration. Therefore the cohomology ring $H^*(F(X,n); \mathbf{Q})$ is isomorphic to the $E_{2l+1} = E_\infty$ ring in the Leray spectral sequence for $f : F(X,n) \hookrightarrow X^n$.

QED.

5 Summary, for smooth projective varieties

In this section, all cohomology will be with rational coefficients. For $a \neq b \in \{1, \ldots, n\}$, let $p_a : H^*(X) \to H^*(X^n)$ and $p^*_{ab} : H^*(X^2) \to H^*(X^n)$ be the obvious pullbacks. For X a smooth projective variety of dimension l over $\mathbf{C}$, let $\Delta \in H^{2l}(X^2)$ denote the class of the diagonal.

Theorem 4 *Let X be a smooth projective variety of dimension l over $\mathbf{C}$. Then the rational cohomology ring of the configuration space $F(X,n)$ is the cohomology of the following differential graded algebra $E(n)$.*

$E(n)$ is the quotient of the graded-commutative $\mathbf{Q}$-algebra

$$H^*(X^n)[G_{ab}],$$

where the G_{ab} are generators of degree $2l-1$ for $1 \le a,b \le n$, $a \ne b$, modulo the following relations:

$$\begin{array}{llcl}
(0) & G_{ab} & = & G_{ba} \\
(1) & G_{ab}G_{ac} + G_{bc}G_{ba} + G_{ca}G_{cb} & = & 0 \text{ for } a,b,c \text{ distinct} \\
(2) & p_a^*(x)G_{ab} & = & p_b^*(x)G_{ab} \text{ for } a \ne b, x \in H^*X.
\end{array}$$

The differential is given by

$$dG_{ab} = p_{ab}^*\Delta.$$

In particular, the rational cohomology ring of $F(X,n)$ is determined by the ring H^*X.

Proof. This follows from theorems 1–3, since $E(n)$ is precisely the E_2 term of the Leray spectral sequence for the inclusion $F(X,n) \hookrightarrow X^n$, as described in theorem 1. We don't have to mention the relation $G_{ab}^2 = 0$, since that follows from graded commutativity in this case. Here the algebra $E(n)$ is determined by the cohomology ring of X, and the differential on $E(n)$ is determined by the class of the diagonal in $H^*(X^2)$, which depends on the cohomology ring of X and on the orientation class in $H^{2l}X$.

If we replace the orientation class by a different generator of $H^{2l}(X,\mathbf{Q}) \cong \mathbf{Q}$, the resulting differential graded algebra $E(n)'$ is the same algebra as $E(n)$, with the differential d multiplied by a nonzero rational constant c. There is an S_n-equivariant isomorphism of differential graded algebras $E(n) \xrightarrow{\cong} E(n)'$ which is the identity on $H^*(X^n)$ and maps G_{ab} to G_{ab}/c. So the rational cohomology ring of $F(X,n)$ (as an S_n-algebra) depends only on the rational cohomology ring of X and not on the orientation class of X. QED.

Theorem 5 *The symmetric group S_n acts on $F(X,n)$ in a natural way. The corresponding action of S_n on the ring $H^*F(X,n) = H^*E(n)$ comes from the following action of S_n on the differential graded algebra $E(n)$: S_n acts on $H^*(X^n) \subset E(n)$ in the obvious way, and*

$$\sigma(G_{ab}) = G_{\sigma(a)\sigma(b)}.$$

Proof. This follows from theorems 1–3, since S_n acts on the Leray spectral sequence for the inclusion $F(X,n) \hookrightarrow X^n$. QED.

Corollary 1 *Let X be a smooth projective variety over $\mathbf{C}$. Let $B(X,n) = F(X,n)/S_n$, the configuration space of n-tuples of distinct unordered points in X. Then the rational cohomology ring of $B(X,n)$ is the cohomology of the differential graded algebra $E(n)^{S_n}$. In particular, the rational cohomology ring of $B(X,n)$ is determined by the ring H^*X.*

Proof. Since the order of the group S_n is invertible in $\mathbf{Q}$, the rational cohomology ring of $B(X,n)$ is just the ring of invariants $H^*(F(X,n);\mathbf{Q})^{S_n}$. Now

$H^*F(X,n)$ is the cohomology of a differential graded algebra $E(n)$, and the S_n-action on $H^*F(X,n)$ comes from an action on $E(n)$. We use the fact that the coefficients are $\mathbf{Q}$ again to show that the cohomology of the ring of invariants $E(n)^{S_n}$ is the ring of invariants in the cohomology of $E(n)$:

$$H^*(E(n)^{S_n}) = (H^*E(n))^{S_n} = (H^*F(X,n))^{S_n} = H^*B(X,n).$$

QED.

Of course, this description of the cohomology ring of $B(X,n)$ is not as explicit as one would like; in particular, one has to work to compute the Betti numbers. Two interesting cases where the Betti numbers have been computed are:

(1) Bödigheimer and Cohen computed the Betti numbers of $B(X,n)$ for all Riemann surfaces X [4]. (There are also some results on the torsion [5].)

(2) Fulton and MacPherson computed that the Betti numbers of $F(X,3)$ are not determined by the Betti numbers of X, for the particular pair of closed manifolds $X = \mathbf{CP}^1 \times \mathbf{CP}^2$ and Y the nontrivial $\mathbf{CP}^1$-bundle $P(O(1) \oplus O(-1))$ over $\mathbf{CP}^2$. We can compute that the same is true for the unordered configuration spaces in this example: X and Y have the same Betti numbers, but $B(X,3)$ and $B(Y,3)$ do not. (By contrast, for odd-dimensional real manifolds X, the Betti numbers of X determine those of $B(X,n)$, and in fact the $\mathbf{Z}/p$-Betti numbers of X determine the $\mathbf{Z}/p$-Betti numbers of $B(X,n)$ [6].)

The calculation is slightly easier using this paper's spectral sequence. The E_∞ term of the spectral sequence computing $H^*(B(X,3),\mathbf{Q})$ has $\mathbf{Q}$-rank as follows:

0	0	0	0						
0	0	0	2	3	2	0			
1	2	5	8	7	3	1	0	0	0

(Here I am only showing the rows which are nonzero in the E_2 term, that is, rows 0, 5, and 10, and only the even columns. Thus the first "2" in the middle row is in row 5, column 6, and so it contributes to $H^{11}(B(X,3),\mathbf{Q})$.) The E_∞ term for $H^*(B(Y,3),\mathbf{Q})$, on the other hand, looks like:

0	0	0	0						
0	0	0	1	3	2	0			
1	2	5	8	7	3	0	0	0	0

So $B(X,3)$ has Betti numbers $b_{11} = 2$, $b_{12} = 1$, while $B(Y,3)$ has Betti numbers $b_{11} = 1$, $b_{12} = 0$. (It is easy to see a priori that the Betti numbers of $F(Y,n)$ and $B(Y,n)$ will be at most the corresponding numbers for $F(X,n)$ and $B(X,n)$ for all n. The point is that the cohomology ring of Y is a deformation of the cohomology ring of X, in the sense that the ring $R_a := Q[u,v]/(u^2-(av)^2, v^3)$ is isomorphic to H^*Y for $a \neq 0$ and to H^*X for $a = 0$. So, in the differential graded algebras $E(n)$ and $E(n)^{S_n}$ constructed from R_a, the linear maps d will have the same rank for all $a \neq 0$, and their ranks for $a = 0$ will be at most that. This implies the statement about Betti numbers.)

References

[1] V. I. Arnold. The cohomology ring of dyed braids. *Mat. Zametki* **5** (1969), 227-231.

[2] L. Avramov. Local algebra and rational homotopy. *Homotopie algébrique et algèbre locale*, 15-43. Astérisque **113-114**, Société Mathématique de France, 1984.

[3] M. Bendersky and S. Gitler. The cohomology of certain function spaces. *Trans. AMS* **326** (1991), 423-440.

[4] C.-F. Bödigheimer and F. Cohen. Rational cohomology of configuration spaces of surfaces. *Algebraic Topology and Transformation Groups* (Göttingen, 1987), 7-13. Lecture Notes in Mathematics **1361**, Springer-Verlag, Berlin–New York, 1988.

[5] C.-F. Bödigheimer, F. Cohen, and R. J. Milgram. Truncated symmetric products and configuration spaces. *Math. Z.* **214** (1993), 179-216.

[6] C.-F. Bödigheimer, F. Cohen, and L. Taylor. On the homology of configuration spaces. *Topology* **28** (1989), 111-123.

[7] F. Cohen. Artin's braid groups, classical homotopy theory, and sundry other curiosities. *Braids*, 167-206. Contemporary Mathematics **78**, American Mathematical Society, Providence, RI, 1988.

[8] F. Cohen and L. Taylor. Gelfand-Fuks cohomology, the cohomology of function spaces, and the cohomology of configuration spaces. *Geometric Applications of Homotopy Theory,* I, 106-143. Lecture Notes in Mathematics **657**, Springer-Verlag, Berlin–New York, 1978.

[9] F. Cohen and L. Taylor. On the representation theory associated to the cohomology of configuration spaces. *Algebraic Topology* (Oaxtepec, 1991), 91-109. Contemporary Mathematics **146**, American Mathematical Society, Providence, RI, 1993.

[10] R. Cohen and U. Tillmann. Lectures on immersion theory. *Differential Geometry and Topology*, Proc. Special Year, Tianjin 1986-87. Lecture Notes in Mathematics **1369**, Springer-Verlag, Berlin–New York, 1989.

[11] P. Deligne. Poids dans la cohomologie des variétés algébriques. *International Congress of Mathematicians*, Vancouver, 1974, 79-85.

[12] E. Fadell and L. Neuwirth. Configuration spaces. *Math. Scand.* **10** (1962), 111-118.

[13] D. B. Fuks. Cohomologies of the braid group mod 2. *Funct. Anal. Appl.* **4** (1970), 143-151.

[14] W. Fulton and R. MacPherson. A compactification of configuration spaces. *Ann. Math.* **139** (1994), 183-225.

[15] M. Goresky and R. MacPherson. *Stratified Morse Theory.* Springer-Verlag, Berlin–New York, 1988.

[16] F. Knudsen. Projectivity of the moduli space of stable curves, II: the stacks $M_{g,n}$. *Math. Scand.* **52** (1983), 1225-1265.

[17] I. Kriz. On the rational homotopy type of configuration spaces. *Ann. Math.* **139** (1994), 227-237.

[18] P. Löffler and J. Milgram. The structure of deleted symmetric products. *Braids*, 415-424. Contemporary Mathematics **78**, American Mathematical Society, Providence, RI, 1988.

Department of Mathematics, University of Chicago, 5734 S. University Ave., Chicago, IL 60637