The cone conjecture for Calabi-Yau pairs in dimension two

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1 Introduction

A central idea of minimal model theory as formulated by Mori is to study algebraic varieties using convex geometry. The cone of curves of a projective variety is defined as the convex cone spanned by the numerical equivalence classes of algebraic curves; the dual cone is the cone of nef line bundles. For Fano varieties (varieties with ample anticanonical bundle), these cones are rational polyhedral by the cone theorem [21, Theorem 3.7]. For more general varieties, these cones are not well understood: they can have infinitely many isolated extremal rays, or they can be "round". Both phenomena occur among Calabi-Yau varieties such as K3 surfaces [22], which can be considered the next simplest varieties after Fano varieties.

The Morrison-Kawamata cone conjecture would give a clear picture of the nef cone for Calabi-Yau varieties [29, 30, 19]. The conjecture says that the action of the automorphism group of the variety on the nef cone has a rational polyhedral fundamental domain. (The conjecture includes an analogous statement about the movable cone; see section 2 for details.) Thus, for Calabi-Yau varieties, the failure of the nef cone to be rational polyhedral is always explained by an infinite discrete group of automorphisms of the variety. It is not clear where these automorphisms should come from. Nonetheless, the conjecture has been proved for Calabi-Yau surfaces by Sterk, Looijenga, and Namikawa [41, 33, 19], the heart of the proof being the Torelli theorem for K3 surfaces of Piatetski-Shapiro and Shafarevich [3, Theorem 11.1]. Kawamata proved the cone conjecture for all 3-dimensional Calabi-Yau fiber spaces over a positive-dimensional base [19]. The conjecture is wide open for Calabi-Yau 3-folds, despite significant results by Oguiso and Peternell [38], Szendröi [43], Uehara [46], and Wilson [48].

The conjecture was generalized from Calabi-Yau varieties to klt Calabi-Yau pairs (X, Δ) in [45]. Here Δ is a divisor on X and "Calabi-Yau" means that $K_X + \Delta$ (rather than K_X) is numerically trivial. In this paper we prove the cone conjecture for all klt Calabi-Yau pairs of dimension 2 (Theorem 4.1), using the geometry of groups acting on convex cones and reduction to the case of K3 surfaces. This is enough to show that the conjecture is reasonable in the greater generality of pairs. More concretely, the theorem gives control over the nef cone and the automorphism group for a large class of rational surfaces, including the Fano surfaces as well as many others. In particular, we get a good description of when the Cox ring (or total coordinate ring) is finitely generated in this class of surfaces (Corollary 5.1).

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2 The cone conjecture

In this section, we state the cone conjecture for klt Calabi-Yau pairs following [45], and discuss some history and examples.

Varieties are irreducible by definition, and a *curve* means a variety of dimension 1. Our main Theorem 4.1 takes the base field to be the complex numbers, but Conjecture 2.1 makes sense over any field. For a projective morphism $f : X \to S$ of normal varieties with connected fibers, define $N^1(X/S)$ as the real vector space spanned by Cartier divisors on X modulo numerical equivalence on curves on X mapped to a point in S (that is, $D_1 \equiv D_2$ if $D_1 \cdot C = D_2 \cdot C$ for all curves C mapped to a point in S). Define a *pseudo-isomorphism* from X_1 to X_2 over S to be a birational map $X_1 \dashrightarrow X_2$ over S which is an isomorphism in codimension one. A *small* **Q***-factorial modification* (SQM) of X over S means a pseudo-isomorphism over S from X to some other **Q**-factorial variety with a projective morphism to S. A Cartier divisor D on X is called *nef* over S (resp. *movable* over S, *effective* over S) if $D \cdot C \geq 0$ for every curve C on X which is mapped to a point in S (resp., if $\operatorname{codim}(\operatorname{supp}(\operatorname{coker}(f^*f_*O_X(D) \to O_X(D)))) \geq 2$, if $f_*O_X(D) \neq 0$).

The canonical divisor is denoted K_X. For an **R**-divisor Δ on a normal **Q**-factorial variety X, the pair (X, Δ) is *klt* if, for all resolutions $\pi : \widetilde{X} \to X$ with a simple normal crossing **R**-divisor $\widetilde{\Delta}$ such that $K_{\widetilde{X}} + \widetilde{\Delta} = \pi^*(K_X + \Delta)$, the coefficients of $\widetilde{\Delta}$ are less than 1 [21, Definition 2.34]. It suffices to check this property on one resolution. For a complex surface X, the pair $(X, 0)$ is klt if and only if X has only quotient singularities [21, Proposition 4.18]. For later use, we define a pair (X, Δ) to be *terminal* if, for all resolutions $\pi : \widetilde{X} \to X$ as above, the coefficients in $\widetilde{\Delta}$ of all exceptional divisors of π are less than 0. (The definition of terminal pairs puts no restriction on the coefficients of Δ itself, although one checks easily (for $\dim(X) \geq 2$) that they are less than 1; that is, a terminal pair is klt.) For a surface X, $(X, 0)$ is terminal if and only if X is smooth.

The *nef cone* $\overline{A}(X/S)$ (resp. the *closed movable cone* $\overline{M}(X/S)$) of X over S is the closed convex cone in $N^1(X/S)$ generated by the numerical classes of divisors that are nef over S (resp. divisors that are movable over S). The *effective cone* $B^e(X/S)$ of X over S is the convex cone, not necessarily closed, generated by Cartier divisors that are effective over S. We call $A^e(X/S) = \overline{A}(X/S) \cap B^e(X/S)$ and $M^e(X/S) = \overline{M}(X/S) \cap B^e(X/S)$ the *nef effective cone* and the *movable effective cone* of X over S, respectively. Finally, a *rational polyhedral* cone in $N^1(X/S)$ means the closed convex cone spanned by a finite set of Cartier divisors on X.

We say that $(X/S, \Delta)$ is a *klt Calabi-Yau pair* if (X, Δ) is a **Q**-factorial klt pair with Δ effective such that $K_X + \Delta$ is numerically trivial over S. (Our main Theorem 4.1 takes S to be a point.) Let $\operatorname{Aut}(X/S, \Delta)$ and $\operatorname{PsAut}(X/S, \Delta)$ denote the groups of automorphisms or pseudo-automorphisms of X over the identity on S that map the divisor Δ to itself.

Conjecture 2.1 *Let $(X/S, \Delta)$ be a klt Calabi-Yau pair.*

(1) The number of $Aut(X/S, \Delta)$-equivalence classes of faces of the cone $A^e(X/S)$ corresponding to birational contractions or fiber space structures is finite. Moreover, there exists a rational polyhedral cone Π which is a fundamental domain for the action of $Aut(X/S, \Delta)$ on $A^e(X/S)$ in the sense that

(a) $A^e(X/S) = \cup_{g \in Aut(X/S,\Delta)} g_*\Pi$,

(b) $Int\,\Pi \cap g_* Int\,\Pi = \emptyset$ *unless* $g_* = 1$.

(2) The number of $PsAut(X/S,\Delta)$*-equivalence classes of chambers* $A^e(X'/S)$ *in the cone* $M^e(X/S)$ *corresponding to marked SQMs* $X' \to S$ *of* $X \to S$ *is finite. Equivalently, the number of isomorphism classes over* S *of SQMs of* X *over* S *(ignoring the birational identification with* X*) is finite. Moreover, there exists a rational polyhedral cone* Π' *which is a fundamental domain for the action of* $PsAut(X/S,\Delta)$ *on* $M^e(X/S)$.

For X terminal and $\Delta = 0$, Conjecture 2.1 is exactly Kawamata's conjecture on Calabi-Yau fiber spaces, generalizing Morrison's conjecture on Calabi-Yau varieties [19, 29, 30]. (The group in part (2) can then be described as $\mathrm{Bir}(X/S)$, since all birational automorphisms of X over S are pseudo-automorphisms when X is terminal and K_X is numerically trivial over S.)

Conjecture 2.1 implies the analogous statement for the group of automorphisms or pseudo-automorphisms of X rather than of (X,Δ). (That slightly weaker formulation of the cone conjecture is used in [45].)

The first statement of part (1) follows from the second statement, on fundamental domains. Indeed, each contraction of X to a projective variety is given by some semi-ample line bundle on X (a line bundle for which some positive multiple is basepoint-free). The class of such a line bundle in $N^1(X)$ lies in the nef effective cone. And two semi-ample line bundles in the interior of the same face of some cone Π determine the same contraction of X, since they have degree zero on the same curves. Thus the second statement of (1) implies the first. We include the first statement in the conjecture because one can try to prove it in some cases where the conjecture on fundamental domains remains open. The first statements of (1) and of (2) are what Kawamata proves for Calabi-Yau fiber spaces of dimension 3 over a positive-dimensional base [19].

For X of dimension at most 2, we only need to consider statement (1), because any pseudo-isomorphism between normal projective surfaces is an isomorphism, and every movable divisor on a surface is nef.

Conjecture 2.1 would fail for Calabi-Yau pairs that are log-canonical (or canonical) rather than klt. Let X be the blow-up of $\mathbf{P}^2$ at 9 very general points. Let Δ be the proper transform of the unique smooth cubic curve through the 9 points; then $K_X + \Delta \equiv 0$, and so (X,Δ) is a canonical Calabi-Yau pair. The surface X contains infinitely many (-1)-curves by Nagata [32], and so the nef cone is not finite polyhedral. But the automorphism group $\mathrm{Aut}(X)$ is trivial [14, Proposition 8] and hence does not have a finite polyhedral fundamental domain on the nef cone. There is also an example of a log-canonical Calabi-Yau surface with rational singularities (with $\Delta = 0$) for which the cone conjecture fails: contract the divisor $R_1 + \cdots + R_4 + 2R_5 \sim -2K_X$ in the surface X of Dolgachev-Zhang [10, Example 6.10].

The conjecture also fails if we allow the $\mathbf{R}$-divisor Δ to have negative coefficients. Let Y be a K3 surface whose nef cone is not finite polyhedral, and let X be the blow-up of Y at a very general point. Let E be the exceptional curve. Then $(X,-E)$ is klt and $K_X - E \equiv 0$. The nef cone of X is not finite polyhedral, but $\mathrm{Aut}(X)$ is trivial.

An interesting class of klt Calabi-Yau pairs are the rational elliptic surfaces (meaning smooth rational surfaces which are minimal elliptic fibrations over $\mathbf{P}^1$).

The cone conjecture was checked for rational elliptic surfaces with no multiple fibers and Mordell-Weil rank 8 by Grassi-Morrison [15, Theorem 2.3], and for all rational elliptic surfaces by [45, Theorem 8.2]. This will be generalized by our main result, Theorem 4.1. Note that the cone conjecture for klt pairs (applied to a suitable divisor Δ with $K_X + \Delta \equiv 0$) describes the whole nef cone of a rational elliptic surface X in $N^1(X)$. By contrast, if we take $\Delta = 0$ and apply the cone conjecture to the elliptic fibration $X \to S$, then we only get information about the relative nef cone in $N^1(X/S)$. For example, the relative nef cone is (trivially) rational polyhedral for any rational elliptic surface, whereas the whole nef cone is rational polyhedral if and only if the Mordell-Weil rank is 0 [45, Theorem 5.2, Theorem 8.2].

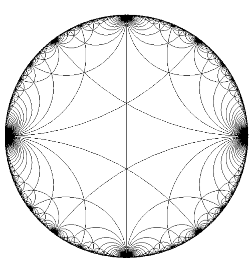
Example. We give an example of a rational surface Y, considered by Zhang [49, Theorem 4.1] and Blache [5, Theorem C(b)(2)], whose nef cone is a 4-dimensional round cone. The surface Y is klt Calabi-Yau, and so the cone conjecture is true by our main Theorem 4.1 (known in this case by Oguiso-Sakurai [39, Corollary 1.9]). Thus the automorphism group of Y must be infinite, and in fact it is a discrete group of isometries of hyperbolic 3-space with quotient of finite volume. (The quotient of hyperbolic 3-space by an index-24 subgroup of the group here is familiar to topologists as the complement of the figure eight knot [25, 1.4.3, 4.7.1].)

Let ζ be a primitive cube root of unity. Let X be the blow-up of $\mathbf{P}^2$ at the 12 points $[1, \zeta^i, \zeta^j], [1, 0, 0], [0, 1, 0], [0, 0, 1]$ over the complex numbers. (This is the dual of the famous Hesse configuration of 9 points lying on 12 lines in the plane [9, 4.6]. Combinatorially, we can identify the Hesse configuration with the 9 points and 12 lines in the affine plane over $\mathbf{Z}/3$.) Let $C_1, \ldots, C_9$ be the proper transforms of the 9 lines through quadruples of the 12 points; these curves have self-intersection -3 in X, and $(X, (1/3)\sum C_i)$ is a klt Calabi-Yau pair. We can contract the 9 disjoint curves E_i to obtain a klt Calabi-Yau surface Y with 9 singular points of type $(1/3)(1, 1)$. Then Y has Picard number 4, and the point of this example is that the nef cone of Y is a round cone (one of the two pieces of $\{x \in N^1(Y) : x^2 \geq 0\}$). That follows by viewing Y as the quotient of an abelian surface $E \times E$ by $\mathbf{Z}/3$; here E is the elliptic curve $\mathbf{C}/\mathbf{Z}[\zeta]$ and $\mathbf{Z}/3$ acts by (ζ, ζ) on $E \times E$. The nef cone of an abelian surface is always round, and in this case $\mathbf{Z}/3$ acts trivially on $N^1(E \times E)$ so that the nef cone of Y is equal to that of $E \times E$.

To prove that Y is isomorphic to the quotient $Y_0 := (E \times E)/(\mathbf{Z}/3)$, first note that the fixed point set of the automorphism group $\mathbf{Z}/3$ of E is a subgroup of order 3. So the fixed point set of $\mathbf{Z}/3$ acting on $E \times E$ is a subgroup of order 9 isomorphic to $(\mathbf{Z}/3)^2$, and these give the 9 singular points of Y_0, all of type $(1/3)(1, 1)$. We can write down 12 curves isomorphic to E on $E \times E$ which go through triples of these 9 points: the curves $E \times 0$, $0 \times E$, the diagonal $\{(x, x)\}$, and $\{(x, -x)\}$, and their translates by the points $(\mathbf{Z}/3)^2 \subset E \times E$. The images $E_1, \ldots, E_{12}$ of these curves in Y_0 are isomorphic to $E/(\mathbf{Z}/3)$ (the quotient by the automorphism group) and hence to $\mathbf{P}^1$. Since $\mathbf{Z}/3$ acts freely in codimension 1 on $E \times E$, the quotient Y_0 has $K_{Y_0} \equiv 0$. It follows that the minimal resolution X_0 of Y_0 has $-K_{X_0} \equiv (1/3)\sum_{i=1}^{9} C_i$, where $C_1, \ldots, C_9$ are the (-3)-curves in X_0 contracted in Y_0. The proper transforms of the 12 curves $E_1, \ldots, E_{12}$ on the minimal resolution X_0 of Y_0 have $-K_{X_0} \cdot E_j = 1$ (since each curve E_j meets 3 curves C_i transversely in one point), and so the curves E_j are (-1)-curves. The (-1)-curves $E_1, \ldots, E_{12}$ are disjoint on X_0, and so we can contract them all and get a smooth surface P_0.

The surface $E \times E$ has Picard number 4 (the maximum possible for a complex abelian surface), spanned by the curves $E \times 0$, $0 \times E$, the diagonal $\{(x,x)\}$, and the graph of an order-3 automorphism of E, $\{(x, \zeta x)\}$. Each of these curves is preserved by the action of $\mathbf{Z}/3$ on $E \times E$, and so the quotient Y_0 also has Picard number 4. Therefore the minimal resolution X_0 of Y_0 has Picard number $4 + 9 = 13$, and P_0 has Picard number $13 - 12 = 1$. We have $-K_{P_0} = (1/3)\sum_{i=1}^{9} C_i$, where $C_1, \ldots C_9$ are the images in P_0 of the curves C_i on X_0. Since $-K_{P_0}$ is effective and not zero while P_0 has Picard number 1, $-K_{P_0}$ must be ample. But the only smooth complex Fano surface with Picard number 1 is $\mathbf{P}^2$, and so P_0 is isomorphic to $\mathbf{P}^2$. We compute that $-K_{P_0} \cdot C_i = 3$ for each curve C_i; since $-K_{\mathbf{P}^2} = O(3)$, this says that $C_1, \ldots, C_9$ are lines in $\mathbf{P}^2$. We have a set of 9 lines and 12 points (the images of the curves E_i on X_0) in the complex projective plane such that every line goes through 4 points and every point lies on 3 lines. One can check by hand that the only such arrangement, modulo automorphisms of the plane, is the dual Hesse arrangement mentioned above. Thus we have identified the blow-up X of $\mathbf{P}^2$ defined earlier with the minimal resolution of $(E \times E)/(\mathbf{Z}/3)$, as promised.

The cone conjecture, a theorem in this case, implies that $\mathrm{Aut}(Y)$ must be a discrete group acting on hyperbolic 3-space (the ample cone of Y modulo scalars) with finite-volume quotient (since every finite polytope in hyperbolic space has finite volume, even if its vertices are at infinity). In this example, the endomorphism ring of the elliptic curve E is $\mathbf{Z}[\zeta]$, and so the endomorphism ring of $E \times E$ is the ring of 2×2 matrices $M_2(\mathbf{Z}[\zeta])$. So the automorphism group of $E \times E$, fixing the origin, is $GL(2, \mathbf{Z}[\zeta])$; the automorphism group of $E \times E$ as a surface is this group together with translations. Any automorphism of $Y = (E \times E)/(\mathbf{Z}/3)$ lifts to an automorphism of $E \times E$ as a surface, since $E \times E$ is the "index-one cover" of Y (defined in section 3). We deduce that $\mathrm{Aut}(Y)$ is the group $(GL(2, \mathbf{Z}[\zeta])/(\mathbf{Z}/3)) \ltimes (\mathbf{Z}/3)^2$, and its image $\mathrm{Aut}^*(Y)$ in $GL(N^1(Y))$ is $PGL(2, \mathbf{Z}[\zeta])$. A decomposition of hyperbolic 3-space into fundamental domains for this group looks roughly like the figure (an analogous picture in the hyperbolic plane).



In particular, any choice of rational polyhedral fundamental domain has a vertex at the boundary of hyperbolic space as in the figure, because there are rational points on the boundary of the nef cone (corresponding to elliptic fibrations of Y). The automorphism group of the smooth rational surface X is the same as that of Y, but now acting on hyperbolic space of dimension $\rho(X) - 1 = 12$. (Clearly every automorphism of Y lifts to the minimal resolution X; the converse holds because $-K_X \equiv (1/3)\sum C_i$, and so the 9 (-3)-curves C_i on X which are contracted in Y are intrinsically picked out as the only curves on which $-K_X$ has negative degree.) The nef cone of X modulo scalars is not all of hyperbolic space, because X contains curves of negative self-intersection (including infinitely many (-1)-curves). The cone conjecture for X is not immediate from that for Y, but it also follows from Theorem 4.1.

Another feature of this example is that $PGL(2, \mathbf{Z}[\zeta])$ is the unique non-cocompact group of orientation-preserving isometries of hyperbolic 3-space of minimum covolume (about 0.085) [26].

3 Klt Calabi-Yau surfaces

In this section, we prove the cone conjecture for klt Calabi-Yau surfaces (as opposed to pairs), Theorem 3.3. This result was stated by Suzuki [42]. Suzuki's ideas were inspiring for this paper, but the proof there is incomplete.

To see the difficulty, let Y be a K3 surface with a node. Then there are two possible types of curves on Y with negative self-intersection, those with self-intersection -2 disjoint from the node and those with self-intersection $-3/2$. (If $X \to Y$ is the minimal resolution with exceptional curve E, the second type of curve is the image of a (-2)-curve on X that meets E transversely in one point.) The second type is missing in [42] (see the definition of the set N' and the reflection group Γ). This makes a difference because the reflection in a $(-3/2)$-curve does not preserve the $\mathbf{Z}$-lattice $S = \{x \in \mathrm{Pic}(X) : x \cdot E = 0\}$. Worse, the angle between two such reflections need not be a rational multiple of π (take $(-3/2)$-curves C_1 and C_2 through the same node of Y with $C_1 \cdot C_2 = 1/2$; this is what happens if the proper transforms of C_1 and C_2 are disjoint on the minimal resolution X). So the group generated by reflections in $(-3/2)$-curves need not be discrete in $GL(S_{\mathbf{R}})$. As a result, the nef cone of Y need not be a Weyl chamber for any reflection group acting on the positive cone. So Lemma 2.4 and Proposition 2.5 in [42] do not work.

Our proof of Theorem 3.3, applied to the example of a K3 surface with a node, works instead by reducing to the minimal resolution. The first version of our proof used hyperbolic geometry to make this reduction, but we can now use a simple general result by Looijenga, Theorem 3.1 below. (Under the same hypothesis as in Theorem 3.1, plus the assumption that the group preserves a bilinear form of signature $(1, *)$, our proof constructed a fundamental domain as a "Dirichlet domain." That assumption holds for the automorphism group of a projective surface Y acting on $N^1(Y)$; as a result, such a group can be viewed as a group of isometries of real hyperbolic space. Looijenga constructs a fundamental domain using the dual vector space, thereby avoiding any need for the given group representation to be self-dual. His fundamental domain coincides with a Dirichlet domain when the representation preserves a bilinear form of signature $(1, *)$.)

Theorem 3.1 *(Looijenga [24, Proposition 4.1, Application 4.15]) Let S be a finitely generated free $\mathbf{Z}$-module and A a closed strictly convex cone in $S_{\mathbf{R}}$ with nonempty interior. Let G be a subgroup of $GL(S)$ which preserves the cone A. Suppose that there is a rational polyhedral cone $C \subset A$ such that $\cup_{g \in G} gC$ contains the interior of A. Then $\cup_{g \in G} gC$ is equal to the convex hull A^+ of the rational points in A, and G has a rational polyhedral fundamental domain on A^+. That is, there is a rational polyhedral cone Π such that $\cup_{g \in G} g\Pi = A^+$ and $\mathrm{Int}\,\Pi \cap g(\mathrm{Int}(\Pi)) = \emptyset$ unless $g = 1$.*

We first note the following consequence of the abundance theorem in dimension 2.

Lemma 3.2 *Let (X, Δ) be a klt Calabi-Yau pair of dimension 2. Then any nef effective $\mathbf{R}$-divisor on X is semi-ample.*

Proof. Since (X, Δ) is a klt pair and $K_X + \Delta$ is nef (being numerically trivial), $K_X + \Delta$ is semi-ample by the abundance theorem in dimension 2 [12, 11]. Therefore $K_X + \Delta$ is $\mathbf{R}$-linearly equivalent to zero. Next, for any nef effective $\mathbf{R}$-divisor D,

$(X, \Delta + \epsilon D)$ is a klt pair for $\epsilon > 0$ small, and $K_X + \Delta + \epsilon D$ is nef. By abundance again, $K_X + \Delta + \epsilon D$ is semi-ample. That is, ϵD is semi-ample. QED

Theorem 3.3 *The cone conjecture holds for any klt Calabi-Yau surface.*

Proof. Let Y be a klt Calabi-Yau surface. Let $I = I(Y)$ be the global index of Y, that is, the least positive integer such that IK_Y is Cartier and linearly equivalent to zero, and let $Z = \operatorname{Spec}(\oplus_{i=0}^{I-1} O_Y(-iK_Y)) \rightarrow Y$ be the global index-one cover of Y. Then Z is a surface with Du Val singularities that has trivial canonical bundle, and Y is the quotient of Z by an action of $\mathbf{Z}/I$. Let M be the minimal resolution of Z. The smooth surface M has trivial canonical bundle and hence is a K3 surface or abelian surface. By uniqueness of the minimal resolution, $\mathbf{Z}/I$ acts on M; let X be the quotient surface.

$$\begin{array}{ccc} M & \xrightarrow[\mathbf{Z}/I]{} & X \\ \downarrow & & \downarrow \\ Z & \xrightarrow[\mathbf{Z}/I]{} & Y \end{array}$$

By Sterk-Looijenga-Namikawa, we know the cone conjecture for the smooth Calabi-Yau surface M [41], [19, Theorem 2.1]. Oguiso-Sakurai proved the cone conjecture for the quotient of a smooth Calabi-Yau surface by a finite group action, in particular for $X = M/(\mathbf{Z}/I)$ [39, Corollary 1.9]. That is, there is a rational polyhedral cone $B \subset A^e(X)$ which is a fundamental domain for the action of $\operatorname{Aut}(X)$ on the nef effective cone $A^e(X)$. The theorem follows from Lemma 3.4, where we take $\Delta = 0$. QED

Lemma 3.4 *Let $X \rightarrow Y$ be a proper birational morphism of klt surfaces. Let Δ be an $\mathbf{R}$-divisor on X and Δ_Y its pushforward to Y. If $Aut(X, \Delta)$ has a rational polyhedral fundamental domain on the nef effective cone of X, then $Aut(Y, \Delta_Y)$ has a rational polyhedral fundamental domain on the nef effective cone of Y.*

Proof. The cone of curves $\overline{Curv}(X)$ is defined as the convex cone spanned by the classes of curves in $N_1(X) = N^1(X)^*$. Let F_0 be the face of $\overline{Curv}(X)$ spanned by the curves in X that map to a point in Y. Then the nef cone $\overline{A}(X)$ has nonnegative pairing with F_0, and the nef cone of Y is $\overline{A}(Y) = \overline{A}(X) \cap F_0^{\perp}$; thus $\overline{A}(Y)$ is a face of $\overline{A}(X)$. Likewise, the nef effective cone of Y is $A^e(Y) = A^e(X) \cap F_0^{\perp}$, as one immediately checks. (In one direction, the image in Y of an effective divisor on X is effective; in the other, the pullback to X of an effective $\mathbf{Q}$-divisor on Y is effective.)

The subgroup H of $G = \operatorname{Aut}(X, \Delta)$ that maps the face F_0 of curves contracted by $X \rightarrow Y$ into itself is a subgroup of $\operatorname{Aut}(Y, \Delta_Y)$. Equivalently, H is the subgroup of G that maps the face $\overline{A}(Y)$ of $\overline{A}(X)$ into itself. If we prove the cone conjecture for this subgroup of $\operatorname{Aut}(Y, \Delta_Y)$, the statement for the whole group $\operatorname{Aut}(Y, \Delta_Y)$ follows.

We know that there is a rational polyhedral fundamental domain B for G acting on $A^e(X)$, and so $A^e(X) = \cup_{g \in G}\, gB$. It follows that $A^e(Y) = \cup_{g \in G}\, gB \cap F_0^{\perp}$. Here each set $gB \cap F_0^{\perp}$ is a rational polyhedral cone contained in $A^e(Y)$.

We will show that these cones fall into finitely many orbits under $H \subset \operatorname{Aut}(Y, \Delta_Y)$. For an element g of G, $gB \cap F_0^{\perp}$ is a face of gB (possibly just 0). So we can divide the nonzero intersections $gB \cap F_0^{\perp}$ into finitely many classes corresponding to the faces B_i of B such that $gB \cap F_0^{\perp} = gB_i$. Fix the face B_i of B (as we can, because the rational polyhedral cone B has only finitely many faces). If $B_i = 0$, then all the cones gB_i are equal to 0 and so they form a single H-orbit. So we can assume that the face B_i of B is not 0.

Consider the contraction $X \to Z$ given by a $\mathbf{Q}$-divisor in the interior of B_i. This makes sense because B_i is contained in the nef effective cone of X and every nef effective $\mathbf{Q}$-divisor on X is semi-ample by Lemma 3.2. Here Z is not a point, because $B_i \neq 0$. Since $X \to Z$ has fiber dimension at most 1, there are only finitely many numerical equivalence classes of curves in X contracted by $X \to Z$. Therefore there are only finitely many intermediate contractions $X \to Y_j \to Z$, say $1 \leq j \leq r$. For all the cones gB_i with $g \in G$ that are contained in $F_0^{\perp}$ (the nef cone of Y), the automorphism g of X moves the contraction $X \to Z$ into one that factors through Y; so g moves the contraction $X \to Y_j$ to $X \to Y$ for some $1 \leq j \leq r$. Fix one value of j, and consider only those cones gB_i such that g moves the contraction $X \to Y_j$ to $X \to Y$. I claim that these cones form only a single orbit under the group H. Indeed, for two cones g_1B_i and g_2B_i with the properties mentioned, the automorphism $g_2g_1^{-1}$ of X moves the cone g_1B_i to the cone g_2B_i, and it preserves the contraction $X \to Y$, which means that it belongs to the subgroup H of G.

Thus $A^e(Y)$ is the union of the rational polyhedral cones $gB \cap F_0^{\perp}$, and these cones fall into finitely many orbits under $H \subset \mathrm{Aut}(Y, \Delta_Y)$. Taking the convex hull of finitely many of these cones, we find a rational polyhedral cone C inside $A^e(Y)$ such that $A^e(Y) = \cup_{g \in H} gC$. By Theorem 3.1, this statement implies the cone conjecture for (Y, Δ_Y). QED

4 Klt Calabi-Yau pairs of dimension 2

Theorem 4.1 *Let (X, Δ) be a klt Calabi-Yau pair of dimension 2 over the complex numbers. Then Conjecture 2.1 is true. That is, the action of Aut(X, Δ) on the nef effective cone has a rational polyhedral fundamental domain. As a result, the number of Aut(X, Δ)-equivalence classes of faces of the nef effective cone corresponding to birational contractions or fiber space structures is finite.*

We remark that Harbourne's "K3-like rational surfaces" have similar finiteness properties [16], although they never have a divisor Δ with (X, Δ) klt Calabi-Yau.

Proof of Theorem 4.1. Since (X, Δ) is a klt pair, it has a terminal model $(\widetilde{X}, \widetilde{\Delta})$ [4, Corollary 1.4.3]. That is, we have a birational projective morphism $\pi : \widetilde{X} \to X$, $\widetilde{\Delta}$ is effective, $K_{\widetilde{X}} + \widetilde{\Delta} = \pi^*(K_X + \Delta)$, and $(\widetilde{X}, \widetilde{\Delta})$ is terminal. (Informally, the terminal model of (X, Δ) is the maximum blow-up of X such that $\widetilde{\Delta}$ is effective.) By our assumptions, $(\widetilde{X}, \widetilde{\Delta})$ is a terminal Calabi-Yau pair of dimension 2, and in particular $\widetilde{X}$ is smooth. The cone conjecture for $(\widetilde{X}, \widetilde{\Delta})$ implies it for (X, Δ), by Lemma 3.4. So we can assume that X is smooth and (X, Δ) is a terminal Calabi-Yau pair.

It is straightforward from the definition that any two terminal models of a pair (X, Δ) are isomorphic in codimension 1. As a result, the terminal model of a 2-dimensional pair is unique.

Example. The terminal model of a pair $(X, 0)$ of dimension 2 may involve more blowing up than the usual minimal resolution of X. For a smooth surface X

and a divisor Δ consisting of two smooth curves with coefficients a and b that meet transversely at a point p, the pair (X, Δ) is klt if and only if $a < 1$ and $b < 1$. Let $\widetilde{X}$ be the blow-up of X at p; then the coefficient in $\widetilde{\Delta}$ of the exceptional curve E is $a + b - 1$. Therefore the terminal model of (X, Δ) will involve blowing up the point p exactly when $a + b - 1 \geq 0$. In fact, if a and b are close to 1, then $a + b - 1$ is also close to 1, although slightly smaller. So the terminal model of (X, Δ) may involve arbitrarily many blow-ups, depending on how close the coefficients a and b are to 1.

We are given a terminal Calabi-Yau pair (X, Δ). If $\Delta = 0$, then X is a smooth Calabi-Yau surface and we know the cone conjecture by Sterk-Looijenga-Namikawa [41, 19]. So we can assume that $\Delta \neq 0$. Using the minimal model program for surfaces, Nikulin showed that X is either rational or a $\mathbf{P}^1$-bundle over an elliptic curve with Δ nef [35, 4.2.1], [1, Lemma 1.4]. In the latter case, the nef effective cone is rational polyhedral in $N^1(X) \cong \mathbf{R}^2$, spanned by a fiber of the $\mathbf{P}^1$-bundle together with Δ (which gives an elliptic fibration of X). Thus the cone conjecture is true for X.

Thus, from now on, we can assume that the smooth projective surface X is rational. One consequence is that $\mathrm{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{R} = N^1(X)$; that is, we need not distinguish between linear and numerical equivalence on X. We also deduce that rational points in the nef cone are effective, as follows.

Lemma 4.2 *Let X be a smooth projective rational surface with $-nK_X$ effective for some $n > 0$. Let L be a nef line bundle on X. Then L is effective.*

Proof. Since X is a rational variety, the holomorphic Euler characteristic $\chi(X, O)$ is 1. By Riemann-Roch, $\chi(X, L) = (L^2 + L \cdot (-K_X))/2 + 1$. Since L is nef and a multiple of $-K_X$ is effective, we have $\chi(X, L) \geq 1$. (The intersection of two nef divisors on a projective surface is nonnegative, because the nef cone is the closure of the ample cone.) An effective divisor equivalent to $-nK_X$ is nonzero, since X is rational. Since L is nef, it follows that an ample line bundle A has $A \cdot (K_X - L) < 0$. Therefore $h^0(X, K_X - L) = 0$. Thus $h^0(X, L) = h^0(X, L) + h^0(X, K_X - L) \geq \chi(X, L) \geq 1$. QED

If X has Picard number at most 2, then $A^e(X)$ is rational polyhedral and so the cone conjecture is true. (For Picard number 2, since X is rational, it is a $\mathbf{P}^1$-bundle over $\mathbf{P}^1$, $X \cong P(O \oplus O(a))$ for some $a \geq 0$. The nef effective cone is spanned by two semi-ample divisors, corresponding to the projection $X \to \mathbf{P}^1$ and the contraction of the $(-a)$-section.) From now on, we can assume that X has Picard number at least 3. We do this to ensure that every K_X-negative extremal ray in $\overline{Curv}(X)$ is spanned by a (-1)-curve [21, Lemma 1.28].

We are assuming that $K_X + \Delta \equiv 0$, and so $-K_X \equiv \Delta$ is effective. As a result, $-K_X$ has a Zariski decomposition $-K_X = P + N$, meaning that P is a nef $\mathbf{Q}$-divisor class, N is an effective $\mathbf{Q}$-divisor with negative definite intersection pairing among its components, and $P \cdot N = 0$ [2, Theorem 14.14]. I claim that P is semi-ample. Indeed, the properties stated of the Zariski decomposition imply that the effective $\mathbf{R}$-divisor Δ numerically equivalent to $-K_X$ must contain N; that is, the divisor $P := \Delta - N$ is effective. By Lemma 3.2, every nef effective $\mathbf{R}$-divisor on X is semi-ample. Thus P is semi-ample.

In particular, the Iitaka dimension of P is either 0, 1, or 2, and this gives the main division of the proof into cases. (By definition, P has Iitaka dimension r if there is a positive integer m_0 and positive numbers a, b such that $am^r \leq h^0(X, mP) \leq bm^r$ for all positive multiples m of m_0 [18, Chapter 10].) The sections of multiples mP can be identified with the sections of $-mK_X$ by adding mN, when $-mK_X$ and mP are both integral divisors; so we can also describe the three cases as $-K_X$ having Iitaka dimension 0, 1, or 2. The group $\mathrm{Aut}^*(X) = \mathrm{im}(\mathrm{Aut}(X) \to GL(N^1(X)))$ is finite when $-K_X$ has Iitaka dimension 2 and virtually abelian for Iitaka dimension 1, whereas it can be a fairly general group acting on hyperbolic space when the Iitaka dimension is 0.

We start with the easiest case, where $-K_X$ is big (that is, it has Iitaka dimension 2). In this case, we will show that the Cox ring $\mathrm{Cox}(X) \cong \oplus_{L \in \mathrm{Pic}(X)} H^0(X, L)$ is finitely generated, which is stronger than the cone conjecture.

The following argument works in any dimension. Since Δ is big, it is $\mathbf{R}$-linearly equivalent to $A + E$ for an ample $\mathbf{R}$-divisor A and an effective $\mathbf{R}$-divisor E [23, Proposition 2.2.22]. Let $\Gamma = (1-\epsilon)\Delta + \epsilon E$ for $\epsilon > 0$ small. Then Γ is effective, (X, Γ) is klt, and $-(K_X + \Gamma) \equiv \epsilon A$ is ample. That is, (X, Γ) is a klt Fano pair. Birkar-Cascini-Hacon-M^cKernan showed that klt Fano pairs of any dimension have finitely generated Cox ring [4, Corollary 1.3.1], as we want.

In dimension 2, this was known earlier: by the cone theorem, a klt Fano pair (X, Γ) has rational polyhedral cone of curves, and every face of this cone can be contracted [21, Theorem 3.7]. In dimension 2, that is enough (together with $\mathrm{Pic}(X) \otimes \mathbf{R} = N^1(X)$) to imply that the Cox ring is finitely generated [17]. In particular, the nef effective cone is rational polyhedral, and so the cone conjecture is true for X.

Next, suppose that $-K_X$ has Iitaka dimension 1. Then the semi-ample divisor P determines a fibration of X over a curve B, and $P^2 = 0$. We have $-K_X \cdot P = (P + N) \cdot P = 0$, and so the generic fiber of $X \to B$ has genus 1. By repeatedly contracting (-1)-curves contained in the fibers of $X \to B$, we find a factorization $X \to Y \to B$ through a minimal elliptic surface $Y \to B$. (The curves being contracted need not be those in N, as one sees in examples.) Write π for the contraction $X \to Y$ and $\Delta_Y = \pi_*(\Delta)$. Since $K_X + \Delta \equiv 0$, we have $K_X + \Delta = \pi^*(K_Y + \Delta_Y)$. So (Y, Δ_Y) is a klt Calabi-Yau pair and (X, Δ) is the terminal model of (Y, Δ_Y).

We know the cone conjecture for the minimal rational elliptic surface Y [45, Theorem 8.2]. But in general, blowing up a point on a surface can increase the complexity of the nef cone, for example turning a finite polyhedral cone into one which is not finite polyhedral. Rather than reduce to that earlier result, we will go through the argument directly for X.

Since $-K_Y \equiv \Delta_Y$, where K_Y has degree zero on all curves contracted by $Y \to B$, Δ_Y is the sum of some positive real multiples of fibers of $Y \to B$. (Here a fiber means the pullback to Y of a point in B, as a divisor. We are using that the intersection pairing on the curves contained in a fiber is negative semidefinite, with radical spanned by the whole fiber [3, Lemma 8.2].) The Mordell-Weil group of the elliptic fibration $X \to B$ is defined as the group $\mathrm{Pic}^0(X_\eta)$ where X_η is the generic fiber. The Mordell-Weil group acts by birational automorphisms on Y over B, hence by automorphisms of Y since $Y \to B$ is minimal. Since Δ_Y is a sum of some multiples of fibers, the Mordell-Weil group preserves Δ_Y on Y. Since (X, Δ) is the terminal model of (Y, Δ_Y) (and terminal models are unique in dimension 2), the Mordell-Weil group acts by automorphisms of (X, Δ).

The main problem is to show that $\mathrm{Aut}(X, \Delta)$ has finitely many orbits on the set of (-1)-curves in X. To do that, we first show that for each curve C in a fiber of $X \to B$, the intersection number of a (-1)-curve E with C is bounded, independent of E. Indeed, we have $1 = -K_X \cdot E = (P+N) \cdot E$. So, if E is not one of the finitely many curves in N, we have $N \cdot E \geq 0$ and hence $P \cdot E \leq 1$. Since each fiber of $X \to B$ is numerically equivalent to a multiple of P, this gives a bound for $E \cdot C$ for each curve C contained in a fiber of $X \to B$, as we want.

The Picard group $\mathrm{Pic}(X_\eta)$ is the quotient of $\mathrm{Pic}(X)$ by some class aP, $a > 0$, together with all the curves $C_1, \ldots, C_r$ in reducible fibers of $X \to B$. (Indeed, if a fiber contains only one curve C, then the class of C in $\mathrm{Pic}(X)$ is some positive multiple of P.) The degree of a line bundle on X on a general fiber of $X \to B$ is given by the intersection number with bP, for some $b > 0$, and so the Mordell-Weil group $G := \mathrm{Pic}^0(X_\eta)$ is the subquotient of $\mathrm{Pic}(X)$ given by

$$G = P^\perp/(aP, C_1, \ldots, C_r).$$

An element x of the group G acts by a translation on the curve X_η of genus 1, which extends to an automorphism of X as we have shown. This gives an action of G on $\mathrm{Pic}(X)$. We know how translation by an element x of $\mathrm{Pic}^0(X_\eta)$ acts on $\mathrm{Pic}(X_\eta)$: by $\varphi_x(y) = y + \deg(y)x$. Since $bP \in \mathrm{Pic}(X)$ is the class of a general fiber of $X \to B$, this means that φ_x acts on $\mathrm{Pic}(X)$ by

$$\varphi_x(y) = y + (y \cdot bP)x \pmod{aP, C_1, \ldots, C_r}.$$

For an element $x \in P^\perp$ with $x \cdot C_i = 0$ for all the curves C_i, I claim that the automorphism φ_x gives the identity permutation of the curves $C_1, \ldots, C_r$ in the fiber of X over any point $0 \in B$. To prove this, we can replace B by $\mathrm{Spec}(R)$ where R is the henselian local ring of B at 0 (that is, $\mathrm{Spec}(R)$ is the limit of all etale open neighborhoods of 0 in B). The group $\ker(\mathrm{Pic}(X_R) \to \mathbf{Z})$ (where the homomorphism gives the degree of a line bundle on the generic fiber) acts by translations on the generic fiber, hence by automorphisms of the minimal elliptic surface Y_R, and finally (since (X_R, Δ) is the terminal model of (Y_R, Δ_Y) where Δ_Y is some nonnegative real multiple of the whole special fiber) by automorphisms of X_R. The proper base change theorem for etale cohomology [27, Corollary VI.2.5] gives that $H^2(X_R, \mu_m) \cong H^2(X_0, \mu_m)$ for any positive integer m, and this group is $\oplus_{i=1}^r \mathbf{Z}/m$. Since the given element $x \in \mathrm{Pic}(X_R)$ has degree zero on each curve C_i, it maps to zero in $H^2(X_R, \mu_m)$. By the Kummer exact sequence

$$\mathrm{Pic}(X_R) \xrightarrow[m]{} \mathrm{Pic}(X_R) \longrightarrow H^2(X_R, \mu_m),$$

x is divisible in $\mathrm{Pic}(X_R)$, and more precisely in the kernel of the homomorphism $\mathrm{Pic}(X_R) \to \mathbf{Z}$ given by the degree of a line bundle on the generic fiber of $X_R \to \mathrm{Spec}(R)$. Since the group $\ker(\mathrm{Pic}(X_R) \to \mathbf{Z})$ acts by automorphisms on X_R, it permutes the curves $C_1, \ldots, C_r$. The divisibility we proved implies that x induces the identity permutation of $C_1, \ldots, C_r$, as we want.

Using that the action of G on $\mathrm{Pic}(X)$ preserves the intersection product, we deduce that φ_x acts on $\mathrm{Pic}(X)$ by the transformation

$$\varphi_x(y) = y + (y \cdot bP)x - \big[x \cdot y + (1/2)(x \cdot x)(y \cdot bP)\big](bP)$$

for all $x \in P^\perp$ with $x \cdot C_i = 0$ for all the curves C_i, and for all $y \in \mathrm{Pic}(X)$. (In terms of hyperbolic geometry, this formula defines a strictly parabolic transformation.)

Now let E_1 and E_2 be any two (-1)-curves on X not contained in fibers of $X \to B$ such that $E_1 \cdot bP = E_2 \cdot bP$ (write $m = E_1 \cdot bP$), $E_1 \cdot C_i = E_2 \cdot C_i$ for all the curves C_i, and $E_1 \equiv E_2 \pmod{m\text{Pic}(X)}$. Let $x = (E_2 - E_1)/m \in \text{Pic}(X)$. Then x is in $P^{\perp}$, we have $x \cdot C_i = 0$ for all the curves C_i, and

$$\begin{aligned}\varphi_x(E_1) &= E_1 + (E_1 \cdot bP)x - \big[(x \cdot E_1) + (1/2)(x \cdot x)(E_1 \cdot bP)\big](bP) \\ &= E_2.\end{aligned}$$

We have shown that the intersection numbers $m = E \cdot bP$ and $E \cdot C_i$ are bounded, among all (-1)-curves E on X. So, apart from the finitely many (-1)-curves contained in fibers of $X \to B$, the (-1)-curves E are divided into finitely many classes according to m, the intersection numbers of E with the curves C_i, and the class of E in $\text{Pic}(X)/m$. By the previous paragraph, the (-1)-curves on X fall into finitely many orbits under the action of G we defined. This completes the proof that $\text{Aut}(X, \Delta)$ has finitely many orbits on the set of (-1)-curves.

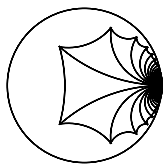
We now describe all the extremal rays of the cone of curves $\overline{Curv}(X)$, following Nikulin [36, Proposition 3.1]. We have mentioned that every K_X-negative extremal ray is spanned by a (-1)-curve. On the other side, a K_X-positive extremal ray must be spanned by one of the finitely many curves in N. (Since $P \cdot N = 0$, the curves in N are contained in fibers of the elliptic fibration $X \to B$ given by P.) Finally, let $\mathbf{R}^{\geq 0}x$ be an extremal ray of $\overline{Curv}(X)$ in $K_X^{\perp}$. Suppose x is not a multiple of a curve in N; then $N \cdot x \geq 0$ and $P \cdot x = 0$. Since $-K_X \equiv P + N$, it follows that $P \cdot x = 0$. Since $P^2 = 0$, the Hodge index theorem gives that x is a multiple of P in $N^1(X)$ or $x^2 < 0$. In the latter case, the ray $\mathbf{R}^{\geq 0}x$ must be spanned by a curve C. Since $P \cdot C = 0$, C is a curve in some fiber of $X \to B$. There are only finitely many numerical equivalence classes of curves in the fibers of $X \to B$. We conclude that almost all (all but finitely many) extremal rays of $\overline{Curv}(X)$ are spanned by (-1)-curves.

Moreover, the only possible limit ray of the (-1)-rays is $\mathbf{R}^{\geq 0}P$. Indeed, if $\mathbf{R}^{\geq 0}x$ is a limit ray of (-1)-rays, then $x^2 = 0$ and $-K_X \cdot x = 0$. (For an ample line bundle A, all (-1)-curves have $E^2 = -1$ and $-K_X \cdot E = -1$, while their degrees $A \cdot E$ in an infinite sequence must tend to infinity. Since $0 < A \cdot x < \infty$, this proves the properties stated of x.) Also, $N \cdot x \geq 0$ since N has nonnegative intersection with almost all (-1)-curves, and $P \cdot x \geq 0$, while $-K_X = P + N$; so $P \cdot x = 0$. Since $P^2 = 0$ and $x^2 = 0$, x is a multiple of P by the Hodge index theorem.

We can deduce that the nef cone $\overline{A}(X)$ is rational polyhedral near any point y in $\overline{A}(X)$ not in the ray $\mathbf{R}^{>0}P$. First, such a point has $y^2 \geq 0$ and also $P \cdot y \geq 0$, since y and P are nef. If $P \cdot y$ were zero, these properties would imply that y was a multiple of P; so we must have $P \cdot y > 0$. Since the only possible limit ray of (-1)-rays is $\mathbf{R}^{\geq 0}P$, there is a neighborhood of y which has positive intersection with almost all (-1)-curves. Since almost all extremal rays of $\overline{Curv}(X)$ are spanned by (-1)-curves, we conclude that the nef cone $\overline{A}(X)$ is rational polyhedral near y, as claimed.

In particular, for each (-1)-curve E not contained in a fiber of $X \to B$, the face $\overline{A}(X) \cap E^{\perp}$ of the nef cone is rational polyhedral, since it does not contain P. So the cone Π_E spanned by P and $\overline{A}(X) \cap E^{\perp}$ is rational polyhedral.

Let x be any nef $\mathbf{R}$-divisor on X. Let c be the maximum real number such that $y := x - cP$ is nef. Then x and y have the same degree on all curves contained in a fiber of $X \to B$. By our list of the extremal rays in $\overline{Curv}(X)$, there must be some (-1)-curve E not contained in a fiber such that $y \in E^{\perp}$. Therefore x is in the cone Π_E. That is, the nef cone $\overline{A}(X)$ is the union of the rational polyhedral cones Π_E, as in the figure. (The positive cone modulo scalars can be viewed canonically as real hyperbolic space, and the figure shows the nef cone modulo scalars as a convex subset of hyperbolic space.)


P

Any rational point x in the nef cone $\overline{A}(X)$ is effective, by Lemma 4.2. So the rational polyhedral cones Π_E are contained in $A^e(X)$, and $A^e(X) = \overline{A}(X)$ is the union of these cones. Since there are only finitely many $\mathrm{Aut}(X, \Delta)$-orbits of (-1)-curves E, Theorem 3.1 proves the cone conjecture for X.

It remains to consider the case where $-K_X$ has Iitaka dimension 0. In the Zariski decomposition $-K_X \equiv \Delta = P + N$, P is numerically trivial, so $-K_X \equiv N$ where N is an effective $\mathbf{Q}$-divisor with negative definite intersection pairing on its irreducible components. In this case, N is the unique effective $\mathbf{R}$-divisor numerically equivalent to $-K_X$, and so the given divisor Δ is equal to N.

We use the following negativity lemma, which is essentially an elementary result on quadratic forms [6, Lemma V.3.5.6].

Lemma 4.3 *(Negativity lemma) Let N be a set of curves on a smooth projective surface on which the intersection pairing is negative definite. Let $-D$ be a linear combination of the curves N_i which has nonnegative intersection with each N_i. Then D is effective. Moreover, the support of D is a union of some connected components of N.*

In our case, we can contract all the curves in N. (There is a positive linear combination $D = \sum a_i N_i$ with $D \cdot N_i = -1$ for all i by Lemma 4.3. The pair $(X, \Delta + \epsilon D)$ is klt for $\epsilon > 0$ small, and $(K_X + \Delta + \epsilon D) \cdot N_i = -\epsilon < 0$ for all i, so we can contract the N_i's by the cone theorem [21, Theorem 3.7].) Write $\pi : X \to Y$ for the resulting contraction. Since $K_X + \Delta \equiv 0$, we have $K_X + \Delta = \pi^*(K_Y)$. So Y is a klt Calabi-Yau surface and (X, Δ) is the terminal model of Y.

We know the cone conjecture for Y by Theorem 3.3. But that does not immediately imply the statement for X. In general, blowing up a point on a surface increases the Picard number and can make the nef cone more complicated, for example turning a finite polyhedral cone into one which is not finite polyhedral. Since (X, Δ) is the terminal model of Y (and terminal models are unique in dimension 2), every automorphism of Y lifts to an automorphism of (X, Δ). Thus it will suffice to show that $\mathrm{Aut}(Y) = \mathrm{Aut}(X, \Delta)$ has a rational polyhedral fundamental domain on the nef effective cone of X.

We can describe all the extremal rays of the cone of curves $\overline{Curv}(X)$, following Nikulin [36, Proposition 3.1]. We have mentioned that every K_X-negative extremal ray is spanned by a (-1)-curve. On the other side, a K_X-positive extremal ray $\mathbf{R}^{\geq 0}x$ must be spanned by one of the finitely many curves in N, since $0 > -K_X \cdot x = N \cdot x$. Finally, let $\mathbf{R}^{\geq 0}x$ be an extremal ray of $\overline{Curv}(X)$ in $K_X^{\perp}$. This ray may be spanned by one of the curves N_i. If it is not, then $x \cdot N_i \geq 0$ for all i. Therefore $0 = -K_X \cdot x = N \cdot x \geq 0$, and so $N_i \cdot x = 0$ for all i. That is, $x = \pi^*(w)$ for some $w \in \overline{Curv}(Y)$.

Let $N_1, \ldots, N_r$ be the irreducible components of N. A (-1)-curve C in X has $1 = -K_X \cdot C = (\sum a_i N_i) \cdot C = \sum a_i \lambda_i$, where $a_1, \ldots, a_r$ are fixed positive numbers and we write $\lambda_i = C \cdot N_i$. As a result, there are only finitely many possibilities for the natural numbers $(\lambda_1, \ldots, \lambda_r)$, for all (-1)-curves on X not among the curves N_i. Call these the finitely many *types* of (-1)-curves on X.

We now describe the nef cone of X. Every divisor class u on X can be written as $\pi^*(y) - \sum b_i N_i$ for some real numbers b_i and some $y \in N^1(Y)$. If u is nef, then y must be nef on Y. Also, u has nonnegative degree on the curves N_i, which says that $(b_1, \ldots, b_r)$ lies in a certain rational polyhedral cone B. The cone B is contained in $[0, \infty)^r$ by the negativity lemma, Lemma 4.3. By our description of the extremal rays of $\overline{Curv}(X)$, a class $u = \pi^*(y) - \sum b_i N_i$ in $N^1(X)$ is nef if and only if y is nef on Y, $(b_1, \ldots, b_r)$ is in the cone B, and u has nonnegative degree on all (-1)-curves not among the curves N_i in X. The last condition says, more explicitly: for each (-1)-curve C not among the curves N_i in X, we must have

$$\begin{aligned} 0 &\leq C \cdot [\pi^*(y) - \sum b_i N_i] \\ &= y \cdot \pi_*(C) - \sum \lambda_i b_i \end{aligned}$$

where we write $\lambda_i = C \cdot N_i$.

Thus, for $u = \pi^*(y) - \sum b_i N_i$ to be nef means that the numbers b_i satisfy the upper bounds $\sum \lambda_i b_i \leq y \cdot \pi_*(C)$ for all (-1)-curves C on X not among the curves N_i, where $(\lambda_1, \ldots, \lambda_r)$ is the type of C. Notice that a (-1)-curve C on X is determined by its type together with the class $\pi_*(C)$ in $N_1(Y)$.

Since $\mathrm{Aut}(X, \Delta) = \mathrm{Aut}(Y)$, the theorem holds if for every rational polyhedral cone $S \subset A^e(Y)$, the inverse image T of S under $\pi_* : A^e(X) \to A^e(Y)$ is rational polyhedral. (Then the inverse images of any decomposition given by the cone conjecture for Y form a decomposition satisfying the cone conjecture for (X, Δ).) Let us first define T to be the inverse image of S in the nef cone $\overline{A}(X)$; at the end we will check that T is actually contained in the nef effective cone. Since $S \setminus 0$ is compact modulo scalars, it suffices to prove that T is rational polyhedral in the inverse image of some neighborhood of each nonzero point in S.

First, let $y_0 \in S$ be a point with $y_0^2 > 0$. We want to show that only finitely many (-1)-curves in X are needed to define the cone T over a neighborhood of y_0 in S. It suffices to show that for each type λ of (-1)-curves on X, there is a finite set Q of (-1)-curves of type λ such that for all y in some neighborhood of y_0, $y \cdot \pi_*(C)$ is minimized among all (-1)-curves C of type λ by one of the curves in Q. The point is that the type of the (-1)-curve determines the rational number $c := \pi_*(C)^2$. (This can be positive, negative, or zero, as examples show.) The intersection pairing on $N_1(Y)$ has signature $(1, *)$ by the Hodge index theorem. Since $y_0^2 > 0$, the intersection of the hyperboloid $\{z \in N_1(Y) : z^2 = c\}$ with $\{z \in N_1(Y) : |z \cdot y_0| \leq M\}$ is compact, for any number M. So there are only finitely many integral classes z in $N_1(Y)$ with $z^2 = c$ and with given bounds on $z \cdot y_0$, and the same finiteness applies for y in some neighborhood of y_0. Thus only finitely many classes $z = \pi_*(C)$, and hence only finitely many (-1)-curves C, can minimize $y \cdot \pi_*(C)$ for any y in a neighborhood of y_0, as we want.

It remains to consider a nonzero point y_0 in S with $y_0^2 = 0$. Since $S \subset A^e(Y)$ is a rational polyhedral cone contained in the positive cone $\{y \in N^1(Y) : y^2 \geq 0, A \cdot y \geq 0\}$, y_0 must belong to an extremal ray of S. Therefore we can scale y_0 to make it an integral point in $N^1(Y)$ (the class of a line bundle on Y). Since y_0 is a nef integral divisor on the klt Calabi-Yau surface Y, it is semi-ample by Lemmas 4.2 and 3.2. Since $y_0^2 = 0$, the corresponding contraction maps Y onto a curve L.

For each point p of Y over which $\pi : X \to Y$ is not an isomorphism, let D be a curve through p which is contained in a fiber of $Y \to L$ (clearly there is such a curve). Let C be the proper transform of D in X. Since C is contained in a singular fiber of $X \to L$, C has negative self-intersection and hence spans an extremal ray of $\overline{Curv}(X)$. Since C is not among the curves N_i and has positive intersection with some curve N_i, our description of the extremal rays of $\overline{Curv}(X)$ shows that C is a (-1)-curve. Thus, for each connected component R of N (corresponding to a point over which $\pi : X \to Y$ is not an isomorphism), there is a (-1)-curve C on X such that $y_0 \cdot \pi_*(C) = 0$ and $\lambda_i = C \cdot N_i$ is positive for some N_i in R.

Moreover, the set Q of (-1)-curves C in X with $y_0 \cdot \pi_*(C) = 0$ is finite, since such a curve must be contained in one of the finitely many singular fibers of $X \to L$. I claim that these finitely many (-1)-curves are enough to define the cone T over a neighborhood of the vertex y_0 in the rational polyhedral cone S. We can view such a neighborhood (up to scalars) as the set of linear combinations $y = y_0 + \sum c_i v_i$, for some nef classes v_i on Y, with $c_i \geq 0$ near zero. Therefore $y \cdot \pi_*(C) \geq y_0 \cdot \pi_*(C)$ for all (-1)-curves C in X. So $y \cdot \pi_*(C)$ is at least 1 for the (-1)-curves C outside the set Q, whereas it is small (for c_i near zero) for C in the set Q. Therefore the inequality $\sum \lambda_i b_i \leq y \cdot \pi_*(C)$ is only needed for the finitely many curves C in Q; that is, T is rational polyhedral over a neighborhood of y_0 in S.

To check this in detail, we have to recall our earlier comment that for each connected component R of N, Q contains a (-1)-curve C with $\lambda_i > 0$ for some N_i in R. This is needed to show that the inequalities for C in Q imply the inequalities for all (-1)-curves C in X. Namely, the inequalities for C in Q imply that b_i is small (assuming y is near y_0) for some N_i in each connected component of N. By the negativity lemma, since $(b_1, \ldots, b_r)$ is in B, it follows that every b_i is small. Indeed, Lemma 4.3 says that if a point in the rational polyhedral cone B has one b_i equal to zero, then b_j is also zero for every N_j in the same connected component as N_i. This implies the same statement for "small" in place of "zero".

Thus the cone $T \subset \overline{A}(X)$ is rational polyhedral. We actually want to know that this rational polyhedral cone is contained in $A^e(X)$. That is the case, by Lemma 4.2 (on a smooth projective rational surface, every nef $\mathbf{Q}$-divisor class is effective). As explained earlier, since $T \subset A^e(X)$ is rational polyhedral, the cone conjecture for (X, Δ) is proved. QED

5 Finite generation of the Cox ring

Corollary 5.1 *Let (X, Δ) be a klt Calabi-Yau pair of dimension 2 over the complex numbers. The following are equivalent:*

(1) The nef effective cone of X is rational polyhedral. (This means in particular that the nef effective cone is closed.)

(2) The nef cone of X is rational polyhedral.

(3) The image of $Aut(X) \to GL(N^1(X))$ is a finite group.

(4) The image of $Aut(X, \Delta) \to GL(N^1(X))$ is a finite group.

If the first Betti number of X is zero (equivalently, if the irregularity of X is *zero), then properties (1) to (4) are equivalent to finite generation of the Cox ring* $Cox(X) \cong \oplus_{L\in Pic(X)} H^0(X,L)$.

For this class of varieties, property (4) is often an easy way to determine whether the Cox ring is finitely generated. For example, for minimal rational elliptic surfaces, property (4) is equivalent to finiteness of the Mordell-Weil group, which can be described in simple geometric terms [45, Theorem 5.2, Theorem 8.2]. The rational elliptic surfaces with finite Mordell-Weil group have been classified by Miranda-Persson [28] and Cossec-Dolgachev [8]. (See Prendergast-Smith [40] for an analogous classification in dimension three.) The K3 surfaces and Enriques surfaces with finite automorphism group were classified by Nikulin, Vinberg, and Kondo [34, 37, 20, 47].

In any dimension, every variety X of *Fano type*, meaning that there is a divisor Γ with (X, Γ) klt Fano, has finitely generated Cox ring, by Birkar-Cascini-Hacon-McKernan [4, Corollary 1.3.1]. The varieties of Fano type form a subclass of the varieties X which have a divisor Δ with (X, Δ) klt Calabi-Yau, namely the subclass with $-K_X$ big. (Compare the proof of Theorem 4.1 in the case where $-K_X$ is big.) For example, the blow-up X of $\mathbf{P}^2$ at any number of points on a conic is of Fano type. Therefore X has finitely generated Cox ring, as Galindo-Monserrat [13, Corollary 3.3], Castravet-Tevelev [7], and Mukai [31] proved by other methods. Recently Testa, Várilly-Alvarado, and Velasco showed that every smooth rational surface with $-K_X$ big has finitely generated Cox ring [44]; Chenyang Xu showed that such a surface need not be of Fano type.

Proof of Corollary 5.1. The closure of the nef effective cone is the nef cone, and so (1) implies (2). The subgroup of $GL(n, \mathbf{Z})$ preserving a rational polyhedral cone that does not contain a line and has nonempty interior is finite, and so (2) implies (3). Clearly (3) implies (4). Theorem 4.1 shows that $\text{Aut}(X, \Delta)$ acts with rational polyhedral fundamental domain on the nef effective cone, and so (4) implies (1).

Suppose that these equivalent conditions hold. Since (X, Δ) is a klt Calabi-Yau pair, every nef effective $\mathbf{Q}$-divisor on X is semi-ample by Lemma 3.2. Another way to say this is that every face of the cone of curves can be contracted. If $b_1(X) = 0$, Hu and Keel showed that (since X has dimension 2) finite generation of the Cox ring is equivalent to the nef cone being rational polyhedral together with every nef divisor being semi-ample [17]. QED

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