Complex varieties with infinite Chow groups modulo 2

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Schoen gave the first examples of smooth complex projective varieties X and prime numbers l for which the Chow group of algebraic cycles modulo l is infinite [21]. In particular, he showed that this occurs for all prime numbers l with $l \equiv 1 \pmod{3}$, with X the product of three copies of the Fermat cubic curve $x^3 + y^3 + z^3 = 0$. This is a fundamental example, showing how far motivic cohomology with finite coefficients can be from etale cohomology, which is finite in this situation. Nonetheless, the restriction on l was frustrating.

Rosenschon and Srinivas then showed that for a very general principally polarized complex abelian 3-fold X, the Chow group $CH^2(X)/l$ is infinite for all prime numbers l at least some (unknown) constant l_0 [18].

In this paper, we show that for a very general principally polarized complex abelian 3-fold X, the Chow group $CH^2(X)/l$ is infinite for *all* prime numbers l (Theorem 3.1). In particular, these are the first examples of smooth complex projective varieties with infinite mod 2 Chow groups. The prime 2 seemed inaccessible for earlier arguments. The mod 2 result also implies that the Witt group $W(X)$ of quadratic bundles is infinite [17], [23, Theorem 1.4]. Again, these are the first complex varieties known to have infinite Witt group.

The method is flexible, and much of it should apply to other classes of varieties. The infiniteness of $CH^2(X)/l$ arises from pulling back Ceresa cycles, as discussed in section 1, by infinitely many different isogenies. A striking feature of the argument is that the analysis of Chow groups modulo l for a complex variety X involves the reduction of X to characteristic l.

Using products $X \times \mathbf{P}^{n-3}$ for any $n \geq 3$, we have similar examples in higher dimensions:

Corollary 0.1. *For each $n \geq 3$, there is a smooth complex projective n-fold X such that $CH^i(X)/l$ is infinite for all $2 \leq i \leq n-1$ and all prime numbers l.*

By taking the product with a very general elliptic curve, we get varieties for which the subgroup of Chow groups killed by l is infinite. This uses Schoen's theorem on exterior product maps on Chow groups [20, Theorem 0.2].

Corollary 0.2. *For each $n \geq 4$, there is a smooth complex projective n-fold X such that $CH^i(X)[l]$ is infinite for all $3 \leq i \leq n-1$ and all prime numbers l.*

The bounds in these corollaries are optimal. In particular, for any smooth complex projective n-fold X and any prime number l, the group $CH^i(X)/l$ is finite if i is 0, 1, or n, and the l-torsion subgroup $CH^i(X)[l]$ is finite if i is 0, 1, 2, or n. The harder cases are finiteness of $CH^n(X)[l]$, by Roitman's theorem [2, Theorem 5.1], and finiteness of $CH^2(X)[l]$, by the Merkurjev-Suslin theorem [14, section 18.4].

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1 Moduli spaces

A property holds for *very general* complex points of a complex variety S if it holds for all points outside a countable union of lower-dimensional closed subvarieties of S. In particular, we can talk about properties of a very general variety in an irreducible family of varieties.

For a curve C of genus ≥ 2 with a rational point p over a field k, the *Ceresa cycle* is the 1-cycle on the Jacobian $J(C)$ given by $C - C^-$. Here C is embedded in $J(C)$ with p mapping to 0, and C^- denotes the image of that curve by the automorphism $x \mapsto -x$ of the Jacobian. The Ceresa cycle is homologically trivial, and Ceresa showed that it is not algebraically equivalent to zero for a very general complex curve C of genus at least 3 [5]. The choice of point p is irrelevant if we only consider the Ceresa cycle modulo algebraic equivalence. Something similar holds in the Chow group of algebraic cycles modulo rational equivalence. Namely, for a curve C over an algebraically closed field and a positive integer m, the choice of p does not affect the Ceresa cycle in $CH_1(J(C))/m$, since the subgroup of the Chow group consisting of cycles algebraically equivalent to zero is divisible [7, Example 19.1.2].

For a positive integer N, let ζ_N be a fixed primitive Nth root of unity. Define a (full) *level N structure* on a principally polarized abelian variety A of dimension g to be a basis $\{u_1, \ldots, u_g, v_1 \ldots, v_g\}$ of the subgroup of A killed by N such that, with respect to the Weil pairing $A[N] \times A[N] \to \mu_N$, we have $\langle u_i, v_i\rangle = \zeta_N$ for all i, $0 = \langle u_i, u_j\rangle = \langle v_i, v_j\rangle$ for all i and j, and $\langle u_i, v_j\rangle = 0$ if $i \neq j$.

Fix a prime number l. Let N be a prime number at least 3 and different from l. Let $X(N)$ be the moduli space of principally polarized abelian varieties of dimension 3 with a full level N structure with respect to ζ_N. Then $X(N)$ is a smooth quasi-projective integral scheme over $\mathbf{Z}[1/N, \zeta_N]$.

Let L be the function field over $\mathbf{Q}(\zeta_N)$ of the moduli space $X(N)$, and let A be the natural abelian variety over L. The main theorem will be that $CH^2(A_{\overline{L}})/l$ is infinite. (We need $N \geq 3$ for L and A to make sense, because the moduli stack $\mathcal{X}(N)$ has nontrivial generic stabilizer when N is 1 or 2. Also, note that the algebraic closure $\overline{L}$ and the abelian variety $A_{\overline{L}}$ are actually independent of the choice of N, up to isomorphism.)

By Lecomte and Suslin, for any variety X over an algebraically closed field F and any algebraically closed extension field E of F, the natural map $CH^2(X)/m \to CH^2(X_E)/m$ is an isomorphism [13, 22]. As a result, showing that $CH^2(A_{\overline{L}})/l$ is infinite will imply that $CH^2(A)/l$ is infinite for a very general principally polarized complex abelian 3-fold A.

Let $M = M(N)$ be the moduli space of curves of genus 3 with a full level N structure on the Jacobian. Then M is a smooth quasi-projective integral scheme over $\mathbf{Z}[1/N, \zeta_N]$. The convenient feature of abelian 3-folds for us is that the Torelli map $M(N) \to X(N)$ is dominant, of degree 2. (This uses that $N \geq 3$. For N equal to 1 or 2, the moduli stack $\mathcal{X}(N)$ has generic stabilizer group of order 2, and the map $M(N) \to X(N)$ of coarse moduli spaces has degree 1.) That is, most principally polarized abelian 3-folds A over an algebraically closed field are Jacobians; but a general curve of genus 3 has trivial automorphism group, whereas a general abelian 3-fold A has automorphism group ± 1.

It may be helpful to say in more detail why $M(N) \to X(N)$ has degree 2 for $N \geq 3$. Let (A, Θ) be a principally polarized abelian 3-fold with a level N structure s. Then the object $(A, \Theta, -s)$ is isomorphic to (A, Θ, s) (so they represent the same point in $X(N)$), because there is an automorphism of A (namely $x \mapsto -x$) that moves one to the other. But for a general curve C of genus 3, there is no automorphism of C that moves a given level N structure s on $J(C)$ to $-s$; so (C, s) and $(C, -s)$ are different points in $M(N)$.

Let E be the function field of $M(N)$. For any finite extension field E_1 of E such that the universal curve C over E has an E_1-rational point p, we can define the Ceresa cycle $y \in CH^2(J(C)_{E_1})$. We are usually concerned only with the class of y in $CH^2(J(C)_{\overline{E}})/l^m$ for a natural number m; that class is independent of the choice of E_1 and p, since two different Ceresa cycles are algebraically equivalent. In fact, the same argument shows that y is fixed by the action of the Galois group $\mathrm{Gal}(\overline{E}/E)$ on $CH^2(J(C)_{\overline{E}})/l^m$.

Since E is a quadratic extension of L, the function field of $X(N)$, we can view y as a class in $CH^2(A_{\overline{L}})/l^m$ for any m. But it is well-defined only up to sign, because of the choice of isomorphism $J(C) \cong A$. As a result, $\mathrm{Gal}(\overline{L}/L)$ acts on y by $gy = y$ if y is in the index-2 subgroup $\mathrm{Gal}(\overline{L}/E)$, and by $gy = -y$ otherwise.

2 The Ceresa cycle modulo a power of any prime number

In this section, we will show that for each prime number l, the "universal" Ceresa cycle (on the Jacobian A of the generic curve of genus 3, extended to an algebraically closed base field) is nonzero in the Chow group modulo some power of l (Proposition 2.2).

The argument works by proving that the universal Ceresa cycle has nonzero image under a suitable l-adic Abel-Jacobi map. The fact that the Ceresa cycle is nonzero moduli some power of l on A over the function field of some moduli space of genus 3 curves follows from the monodromy calculation that Hain used to prove Ceresa's theorem. Then we have the serious problem of showing that the Ceresa cycle remains nonzero modulo some power of l when the base field becomes algebraically closed. For that, we use a result of Bloch and Esnault (Theorem 2.1 below), an application of Bloch and Kato's work on p-adic Hodge theory, along with further monodromy arguments.

For a variety X over an algebraically closed field, the *coniveau* filtration on etale cohomology is defined by: an element x of $H^*(X, \mathbf{Z}/a)$ is in $N^r H^*(X, \mathbf{Z}/a)$ if there is a closed subset Y of codimension at least r in X such that x restricts to zero in $H^*(X - Y, \mathbf{Z}/a)$. We now state Bloch and Esnault's result [1, section 1].

Theorem 2.1. *Let K be a field with a discrete valuation v, and let k be the residue field. Assume that K has characteristic zero and k is perfect of characteristic $l > 0$.*

Let X be a smooth projective variety over K with good ordinary reduction at v, and let Y be the special fiber over k. Assume either that the crystalline cohomology of Y has no torsion or that

$$\dim(X) < (l-1)/\gcd(e, l-1),$$

where e is the absolute ramification degree of K (meaning that $v(K^*) = \mathbf{Z} \cdot (v(l)/e)$). Finally, let m be a natural number such that $H^0(Y, \Omega^m) \neq 0$.

Then $N^1 H^m(X_{\overline{K}}, \mathbf{F}_l) \neq H^m(X_{\overline{K}}, \mathbf{F}_l)$. Equivalently, writing $\overline{K}(X)$ for the function field, the natural map

$$H^m(X_{\overline{K}}, \mathbf{F}_l) \to H^m(\overline{K}(X), \mathbf{F}_l)$$

is not zero.

Classical Hodge theory implies that a complex variety with nonzero differential forms has nontrivial coniveau filtration on rational cohomology. We need Theorem 2.1, however, in order to say anything about the coniveau filtration on mod l cohomology.

We will apply Theorem 2.1 to an abelian variety X with good ordinary reduction. In this case, the special fiber Y is an ordinary abelian variety over k. Every abelian variety over a perfect field of characteristic $l > 0$ has torsion-free crystalline cohomology [10, section 7.1]. So Bloch-Esnault's result applies for all prime numbers l in this case.

Here is the main result of this section.

Proposition 2.2. *Let l be a prime number, and let N be a prime number at least 3 and different from l. Let E be the function field over $\mathbf{Q}(\zeta_N)$ of the moduli space of curves of genus 3 with level N structure, and let C be the universal curve over E. Let $\overline{L}$ be an algebraic closure of E, and let y be the Ceresa cycle in $CH^2(J(C)_{\overline{L}})$ associated to an $\overline{L}$-point of C. Then there is a positive integer c such that $2y$ is not zero in $CH^2(J(C)_{\overline{L}})/l^c$.*

Proof. Fix a prime number l and a prime number $N \geq 3$ different from l. As in section 1, let L be the function field of the moduli space $X(N)$ of principally polarized abelian 3-folds with a level N structure. Let A be the natural abelian 3-fold over L. (Much of what follows works under some conditions for other abelian 3-folds over fields of characteristic zero.)

Let $\Theta \in H^2(A_{\overline{L}}, \mathbf{Q}_l(1))$ be the given polarization. The *primitive part* $PH^3(A_{\overline{L}}, \mathbf{Q}_l(2))$ is the kernel of multiplication by Θ. The hard Lefschetz theorem over $\mathbf{C}$, translated to etale cohomology, gives a direct-sum decomposition

$$H^3(A_{\overline{L}}, \mathbf{Q}_l(2)) = PH^3(A_{\overline{L}}, \mathbf{Q}_l(2)) \oplus \Theta \cdot H^1(A_{\overline{L}}, \mathbf{Q}_l(1))$$

[8, p. 122].

Let v be the discrete valuation on L whose residue field is the moduli space of principally polarized abelian 3-folds over $\mathbf{F}_l(\zeta_N)$ with level N structure. Since the generic abelian 3-fold in characteristic l is ordinary, A has good ordinary reduction at v. By Theorem 2.1, $N^1 H^3(A_{\overline{L}}, \mathbf{Z}/l(2))$ is a proper subgroup of $H^3(A_{\overline{L}}, \mathbf{Z}/l(2))$. (The Tate twist $\mathbf{Z}/l(2) = (\mu_l)^{\otimes 2}$ makes no difference to the statement, since we are considering etale cohomology over an algebraically closed field.) It follows that $N^1H^3(A_{\overline{L}}, \mathbf{Z}/l^r(2))/l \to H^3(A_{\overline{L}}, \mathbf{Z}/l^r(2))/l$ is not surjective for any positive integer r. So the injection

$$B := (\varprojlim_r N^1H^3(A_{\overline{L}}, \mathbf{Z}/l^r(2))) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l \to H^3(A_{\overline{L}}, \mathbf{Q}_l(2))$$

is not surjective. (Note that the subspace B may a priori be bigger than $N^1H^3(A_{\overline{L}}, \mathbf{Q}_l(2))$. That actually happens in some examples over $\overline{\mathbf{F}_p}$, by Schoen [19, after Theorem 0.4]. It would be relatively easy to prove an upper bound for $N^1H^3(A_{\overline{L}}, \mathbf{Q}_l(2))$, but we need Bloch-Esnault's argument in order to prove an upper bound for B.)

The Galois group $\mathrm{Gal}(\overline{L}/L)$ acts on $H^3(A_{\overline{L}}, \mathbf{Q}_l(2))$, preserving the primitive subspace PH^3. The Galois group $\mathrm{Gal}(\overline{L}/L\overline{\mathbf{Q}}) \subset \mathrm{Gal}(\overline{L}/L)$ maps onto a completion of the congruence subgroup $\Gamma(N)$ of $Sp(6, \mathbf{Z})$, which acts on $H^1(A, \mathbf{Q}_l) \cong (\mathbf{Q}_l)^6$ as the standard representation V of the symplectic group. Since $\Gamma(N)$ is Zariski dense in $Sp(6, \mathbf{Q}_l)$ and $PH^3(A_{\overline{L}}, \mathbf{Q}_l(2))$ is the irreducible representation $\Lambda^3(V)/V$ of $Sp(6, \mathbf{Q}_l)$, the representation of $\mathrm{Gal}(\overline{L}/L)$ on $PH^3(A_{\overline{L}}, \mathbf{Q}_l(2))$ is irreducible. The Galois group also preserves the subspace B in the previous paragraph, and it is clear that B contains the subspace $\Theta \cdot H^1$ (since classes in $\Theta \cdot H^1$ are supported on a theta divisor in $A_{\overline{L}}$). The irreducibility together with the previous paragraph's result implies that B is *equal* to $\Theta \cdot H^1$.

It follows that the inverse limit $\varprojlim_r N^1H^3(A_{\overline{L}}, \mathbf{Z}/l^r(2))$, a finitely generated $\mathbf{Z}_l$-submodule of $H^3(A_{\overline{L}}, \mathbf{Z}_l(2))$, contains $\Theta \cdot H^1(A_{\overline{L}}, \mathbf{Z}_l(1))$ as a subgroup of finite index. So there is an $m \geq 0$ such that for all $r \geq 0$, $l^m N^1H^3(A_{\overline{L}}, \mathbf{Z}/l^r(2))$ is contained in $\Theta \cdot H^1(A_{\overline{L}}, \mathbf{Z}/l^r(2))$. (In the case at hand (with Θ a principal polarization), we could take $m = 0$, but we choose to state the argument in a way that would work more generally.)

Let $P \in CH^3(A \times A)$ be a correspondence (with integer coefficients) such that the action of P on $H^3(A_{\overline{L}}, \mathbf{Q}_l(2))$ sends $\Theta \cdot H^1$ to zero and maps the primitive part $PH^3(A_{\overline{L}}, \mathbf{Q}_l(2))$ to itself by an isomorphism. The existence of such a correspondence is part of the Lefschetz standard conjecture, which is a theorem for abelian varieties [12]. (In fact, P can be defined explicitly as a polynomial in divisor classes on $A \times A$ [15, Remark 5.11].) By the previous paragraph, there is an $a \geq 0$ such that $P_* N^1H^3(A_{\overline{L}}, \mathbf{Z}/l^r(2))$ is killed by l^a for all $r \geq 0$.

The Merkurjev-Suslin theorem implies that for all smooth projective varieties X over $\overline{L}$, Bloch's cycle class map

$$CH^2(X_{\overline{L}})[l^\infty] \to H^3(X_{\overline{L}}, \mathbf{Q}_l/\mathbf{Z}_l(2))$$

is injective, with image $N^1H^3(X_{\overline{L}}, \mathbf{Q}_l/\mathbf{Z}_l(2))$ [14, section 18.4]. So the previous paragraph implies that $P_*(CH^2(A_{\overline{L}})[l^\infty])$ is killed by l^a.

Let E be the function field of the moduli space $M(N)$ of curves of genus 3 with level N structure, and let C be the universal curve over E. Then E is a quadratic extension of L. Let y be the Ceresa cycle in $CH^2(A_{E_1})$ associated to a finite extension E_1 of E and an E_1-point of C, as in section 1. We are primarily interested in the image of y in $CH^2(A_{\overline{L}})/l^m$ for natural numbers m, which is independent of the choice of E_1 and p, but which depends up to sign on the choice of isomorphism $J(C) \cong A_E$.

Let $z = 2y$. Then z is a codimension-2 cycle on $A_{\overline{L}}$ which is homologically trivial, meaning that z maps to zero in $H^4(A_{\overline{L}}, \mathbf{Z}_l(2))$. There is an l-adic Abel-Jacobi map for homologically trivial cycles, taking values in continuous Galois cohomology [4, section 1]:

$$CH^2_{\text{hom}}(A_{E_1}) \to H^1(E_1, H^3(A_{\overline{L}}, \mathbf{Z}_l(2))).$$

Following Jannsen, continuous cohomology means the derived functors of the functor $(M_n) \mapsto \varprojlim_n (M_n)^G$ on inverse systems [11].

We now use that the field L is finitely generated over $\mathbf{Q}$. The following result is modeled on Bloch and Esnault [1, Proof of Proposition 4.1].

Lemma 2.3. *The natural map*

$$H^1(L, P_*H^3(A_{\overline{L}}, \mathbf{Z}_l(2))) \to H^1(L', P_*H^3(A_{\overline{L}}, \mathbf{Z}_l(2)))^{Gal(L'/L)}$$

is an isomorphism for all finite Galois extensions L' of L.

Proof. Let $M = P_*H^3(A_{\overline{L}}, \mathbf{Z}_l(2))$, and let $G = \mathrm{Gal}(\overline{L}/L)$. Then M is a finitely generated free $\mathbf{Z}_l$-module on which G acts with nonzero weight m (namely, $m = -1$). (That is, let Y be a scheme of finite type over $\mathbf{Z}$ with fraction field L (in the case at hand, Y is the moduli space $X(N)$). To say that M has weight m means that the eigenvalues of Frobenius on $M \otimes \mathbf{Q}_l$ at all closed points y of Y in some nonempty open subset are algebraic numbers, with all archimedean absolute values equal to q^m, where q is the order of the residue field at y. To prove that, it suffices to take an open subset of Y where A has good reduction, and then apply Deligne's theorem (the Weil conjecture) [6]. Since A is an abelian variety, we could also reduce to the Weil conjecture for H^1, proved by Weil.)

Let $M_n = M/l^n$ for any natural number n. Write $H^i(G, M)$ for continuous cohomology as defined above. Since the groups M_n are finite, the natural map $H^i(G, M) \to \varprojlim_n H^i(G, M_n)$ is an isomorphism for all i [11, equation 2.1]. We want to show that for any open normal subgroup H of G, the natural map

$$H^1(G, M) \to H^1(H, M)^{G/H}$$

is an isomorphism.

The Hochschild-Serre spectral sequence gives an exact sequence, for each n:

$$0 \longrightarrow H^1(G/H, M_n^H) \longrightarrow H^1(G, M_n) \xrightarrow{\alpha_n} H^1(H, M_n)^{G/H} \longrightarrow H^2(G/H, M_n^H).$$

The groups on the left are finite, and so they satisfy the Mittag-Leffler condition as n varies. This implies the exact sequences:

$$0 \to \varprojlim_n H^1(G/H, M_n^H) \to \varprojlim_n H^1(G, M_n) \to \varprojlim_n \mathrm{im}(\alpha_n) \to 0$$

and

$$0 \to \varprojlim_n \mathrm{im}(\alpha_n) \to \varprojlim_n H^1(H, M_n)^{G/H} \to \varprojlim_n H^2(G/H, M_n^H).$$

The groups M_n^H are finite, and so the inverse system M_n^H satisfies Mittag-Leffler. That implies that the continuous cohomology $H^i(G/H, \varprojlim_n M_n^H)$ is computed by the complex of continuous cochains with coefficients in $\varprojlim_n M_n^H$ [11, Theorem 2.2]. But $\varprojlim M_n^H = (\varprojlim M_n)^H = 0$ because M has nonzero weight as an H-module and is torsion free. So $H^i(G/H, \varprojlim_n M_n^H) = 0$ for all i. By the exact sequences above, the map $H^1(G, M) \to H^1(H, M)^{G/H}$ is an isomorphism. □

By section 1, the Ceresa class y and therefore $z = 2y$ are invariant under $\mathrm{Gal}(\overline{L}/E)$ in $CH^2(A_{\overline{L}})/l^m$, for all natural numbers m. By Lemma 2.3, it follows that z has a well-defined class in $H^1(E, H^3(A_{\overline{L}}, \mathbf{Z}_l(2)))$.

Next, we show that $P_* z$ has nonzero image in $H^1(E, H^3(A_{\overline{L}}, \mathbf{Q}_l(2)))$, which is defined to mean the continuous cohomology group above tensored with $\mathbf{Q}_l$ [11, Definition 5.13]. This follows from Hain's proof of Ceresa's theorem. Let F be the direct limit of the function fields of the moduli spaces $M(N')$ over all positive integers N'. Then $\mathrm{Gal}(F/E)$ is a completion of the congruence subgroup $\Gamma(N)$ in $Sp(6, \mathbf{Z})$. It suffices to show that $P_* z$ in $H^1(E, PH^3(A_{\overline{L}}, \mathbf{Q}_l(2)))$ has nonzero restriction to $H^1(F, PH^3(A_{\overline{L}}, \mathbf{Q}_l(2)))$.

The action of the Galois group of E on the cohomology of $A_{\overline{L}}$ factors through $\mathrm{Gal}(F/E)$, and so we are just claiming that $P_* z$ determines a nonzero homomorphism $\mathrm{Gal}(\overline{L}/F) \to PH^3(A_{\overline{L}}, \mathbf{Q}_l(2))$. Here $\mathrm{Gal}(\overline{L}/E\overline{\mathbf{Q}})$ maps onto a completion of the *Torelli group*, the kernel of the homomorphism from the genus 3 mapping class group to $Sp(6, \mathbf{Z})$. By working over $\mathbf{C}$, it suffices to show that the Ceresa class determines a nonzero homomorphism from the Torelli group to $PH^3(A_{\mathbf{C}}, \mathbf{Q})$. (Here the prime number l is irrelevant.) This is exactly what Hain's computation of the normal function of the Ceresa cycle shows [9, proof of Theorem 8.2]. (In fact, the Ceresa cycle gives an isomorphism from the abelianized Torelli group tensor $\mathbf{Q}$ to $PH^3(A_{\mathbf{C}}, \mathbf{Q})$. Johnson had earlier shown that these two groups are isomorphic.)

Remark 2.4. Shou-Wu Zhang strengthened Ceresa's theorem in a certain direction. Namely, B be a smooth projective curve over a field k_0, and let $\mathcal{C} \to B$ be a non-isotrivial family of curves of genus 3 over B with all fibers smooth. (Such families do exist.) Let C be the generic fiber over $k := k_0(B)$. There is a height pairing $CH^2_{\mathrm{hom}}(J(C)) \otimes CH^2_{\mathrm{hom}}(J(C)) \to \mathbf{Z}$ on the cycles homologically equivalent to zero, defined using intersections on the 4-fold $J(\mathcal{C}) \to B$. Suppose that there is a zero-cycle e of degree 1 on X, and let y be the Ceresa cycle in $CH^2_{\mathrm{hom}}(J(C))$ associated to e. Then Zhang shows that the height pairing $\langle y, y \rangle$ is positive [24, Theorem 1.3.1, Corollary 1.3.4, Theorem 1.5.5]. It follows that for every prime number l, there is a positive integer a such that y is not zero in $CH^2(J(C))/l^a$.

This proves the nontriviality of the Ceresa cycle over relatively small function fields, not just over the function field of the whole moduli space of curves of genus 3. However, we would still need arguments as in this paper in order to argue that the Ceresa cycle remains nonzero in $CH^2(J(C)_{\overline{k}})/l^b$ for some b, where the base field is algebraically closed.

By the properties of the correspondence P, $P_* z$ takes values in $H^1(E, P_* H^3(A_{\overline{L}}, \mathbf{Q}_l(2)))$; so $P_* z$ is nonzero in $H^1(E, P_* H^3(A_{\overline{L}}, \mathbf{Z}_l(2)))$. By definition of this continuous cohomology group, we have an exact sequence [11, 3.1]

$$0 \to \varprojlim_r{}^1 H^0(E, P_* H^3(A_{\overline{L}}, \mathbf{Z}/l^r(2))) \to H^1(E, P_* H^3(A_{\overline{L}}, \mathbf{Z}_l(2))) \to \varprojlim_r H^1(E, P_* H^3(A_{\overline{L}}, \mathbf{Z}/l^r(2))) \to 0.$$

Since $H^3(A_{\overline{L}}, \mathbf{Z}/l^r(2))$ is finite for each r, the H^0 groups on the left are finite, and so they satisfy the Mittag-Leffler condition as r varies; so the derived limit $\varprojlim^1$ is zero. That is,

$$H^1(E, P_* H^3(A_{\overline{L}}, \mathbf{Z}_l(2))) \cong \varprojlim_r H^1(E, P_* H^3(A_{\overline{L}}, \mathbf{Z}/l^r(2))).$$

It follows that every nonzero element of $H^1(E, P_*H^3(A_{\overline{L}}, \mathbf{Z}_l(2)))$ is nonzero modulo l^b for some $b \geq 0$. In particular, there is a b such that P_*z is nonzero in $H^1(E, P_*H^3(A_{\overline{L}}, \mathbf{Z}_l(2)))/l^b$.

Assume that there is a cycle w in $CH^2(A_{\overline{L}})$ such that

$$l^{a+b}w = z.$$

Since z is homologically trivial and the cohomology of $A_{\overline{L}}$ is torsion-free, w is homologically trivial. Let E' be a finite Galois extension of E such that the cycle w is defined over E', and consider w as an element of $CH^2_{\text{hom}}(A_{L'})$. For σ in $\text{Gal}(\overline{L}/L)$, we have

$$\begin{aligned} l^{a+b}P_*(w - \sigma(w)) &= P_*(z - \sigma(z)) \\ &= 0. \end{aligned}$$

Since $P_*(CH^2(A_{\overline{L}})[l^\infty])$ is killed by l^a, it follows that $l^a P_*(w - \sigma(w)) = 0$; that is, $l^a P_*(w)$ is fixed by $\text{Gal}(\overline{L}/E)$. By Lemma 2.3, it follows that $l^a P_*(w)$ can be viewed as an element u of $H^1(E, P_*H^3(A_{\overline{L}}, \mathbf{Z}_l(2)))$, and we have

$$l^b u = P_* z$$

in that group. This contradicts that P_*z is nonzero in $H^1(E, P_*H^3(A_{\overline{L}}, \mathbf{Z}_l(2)))/l^b$. Thus there is no element w as above. In other words,

$$z \neq 0 \in CH^2(A_{\overline{L}})/l^{a+b}.$$

Since z is 2 times the Ceresa cycle y, Proposition 2.2 is proved. □

3 Isogenies

Theorem 3.1. *Let A be a very general principally polarized abelian 3-fold over $\mathbf{C}$. Then $CH^2(A)/l$ is infinite for every prime number l.*

Proof. Fix a prime number N at least 3 and different from l. As discussed in section 1, it suffices to show that $CH^2(A_{\overline{L}})/l$ is infinite, where L is the function field of the moduli space $X(N)$ of principally polarized abelian 3-folds with level N structure.

We will imitate the strategy Nori used to show that the Griffiths group tensor $\mathbf{Q}$ has infinite rank for a very general principally polarized abelian 3-fold A [16]. Rosenschon and Srinivas extended Nori's argument to show that $CH^2(A_{\overline{L}})/l$ is infinite for almost all primes l [18].

Namely, $A_{\overline{L}}$ is the Jacobian of a curve, and so we have a Ceresa cycle y on $A_{\overline{L}}$, well-defined up to sign in $CH^2(A_{\overline{L}})/l^m$ for any m, as discussed in section 1. By Proposition 2.2, there is a positive integer c such that $z := 2y$ is nonzero in $CH^2(A_{\overline{L}})/l^c$.

The plan is to consider infinitely many isogenies from A to other principally polarized abelian 3-folds. Pulling the Ceresa cycles back by these isogenies gives infinitely many nonzero elements of $CH^2(A_{\overline{L}})/l^c$. We argue that these elements of $CH^2(A_{\overline{L}})/l^c$ are all different because they all have different actions of the Galois group $\text{Gal}(\overline{L}/L)$. Thus $CH^2(A_{\overline{L}})/l^c$ is infinite, and it follows that $CH^2(A_{\overline{L}})/l$ is infinite.

Lemma 3.2. *Let $f\colon A \to B$ be an isogeny of principally polarized abelian varieties over an algebraically closed field k. If f has degree prime to l, then the pullback $f^*\colon CH^*(B)/l^c \to CH^*(A)/l^c$ is an isomorphism.*

Proof. f_*f^* is multiplication by $\deg(f)$, and so f^* is split injective on $CH^*(A)/l^c$. The composition f_*f^* is the sum of the translates by elements of the finite group $\ker(f)$. These translates act on Chow groups as the identity modulo algebraic equivalence. Since k is algebraically closed, the group of cycles algebraically equivalent to zero is divisible, and so f_*f^* acts as multiplication by $\deg(f)$ on $CH^*(B)/l^c$. Thus f^* is an isomorphism on Chow groups modulo l^c. $\square$

The abelian 3-fold $A_{\overline{L}}$ has many prime-to-Nl isogenies to principally polarized abelian 3-folds. They are all isomorphic to $A_{\overline{L}}$ as schemes (not as schemes over $\overline{L}$). By Lemma 3.2, the pullback of 2 times the Ceresa cycle y under each of these isogenies is nonzero in $CH^2(A_{\overline{L}})/l^c$. We conclude that all the pullbacks of the Ceresa cycle y are not killed by 2 in $CH^2(A_{\overline{L}})/l^c$.

It remains to show that for a suitable infinite family of isogenies, the pullbacks of y are all different in $CH^2(A_{\overline{L}})/l^c$. Let F be the direct limit of the function fields of the moduli spaces $X(M)$ over all positive integers M. Following Nori, we argue that $\mathrm{Gal}(\overline{L}/F)$ acts by different characters $\mathrm{Gal}(\overline{L}/F) \to \pm 1$ on all these pullbacks.

Choose a sequence $r_1, r_2, \ldots$ of elements in $Sp(6,\mathbf{Q})$ which are distinct in the set $Sp(6,\mathbf{Q})\backslash Sp(6,\mathbf{Z})$. We can assume that each r_i is integral (that is, in $Sp(6,\mathbf{Z}_{(p)})$) at primes p dividing Nl. Just as $Sp(6,\mathbf{R})$ acts on the Siegel upper half-space, $Sp(6,\mathbf{Q})$ acts on the inverse limit of the moduli spaces $X(M)$. In particular, $Sp(6,\mathbf{Q})$ acts by automorphisms on the direct limit F of the function fields of $X(M)$.

The center $\{\pm 1\}$ of $Sp(6,\mathbf{Q})$ acts trivially on F, and so we can also think of this as an action ρ_1 of $GSp(6,\mathbf{Q})$ on F, with the center $\mathbf{Q}^*$ acting trivially. Moreover, for any element $g \in M_6(\mathbf{Z}) \cap GSp(6,\mathbf{Q})$, there are positive integers a and M with a commutative diagram

$$\begin{array}{ccc} A(Ma) & \longrightarrow & A(M) \\ \downarrow & & \downarrow \\ X(Ma) & \longrightarrow & X(M), \end{array}$$

where the top map is an isogeny on the fibers. This induces a commutative diagram

$$\begin{array}{ccc} A_F & \xrightarrow{\rho_2(g)} & A_F \\ \downarrow & & \downarrow \\ \mathrm{Spec}(F) & \xrightarrow{\rho_1(g)} & \mathrm{Spec}(F). \end{array}$$

For $N \geq 3$, the map $M(N) \to X(N)$ has degree 2 and is ramified over the closure of the image of the divisor of hyperelliptic curves in $M(N)$. Let D be the corresponding divisor in the Siegel space H. (The level structure is irrelevant to the definition of D; in other words, D is the inverse image of a divisor in the coarse moduli space $X(1)$ of principally polarized abelian 3-folds.) We use the following observation by Nori [16, Lemma]:

Lemma 3.3. *The subgroup of $Sp(6,\mathbf{R})$ that maps $D \subset H$ into itself is equal to $Sp(6,\mathbf{Z})$.*

Proof. The subgroup K of $Sp(6, \mathbf{R})$ that maps D into itself is a closed Lie subgroup of $Sp(6, \mathbf{R})$. The Lie algebra of K is stable under the adjoint action of K, which contains $Sp(6, \mathbf{Z})$, and that is Zariski dense in $Sp(6, \mathbf{R})$. So this Lie algebra is zero or all of $\mathfrak{sp}(6, \mathbf{R})$. In the latter case, K is equal to $Sp(6, \mathbf{R})$, which is false since D is not all of H. So K is discrete. Since $Sp(6, \mathbf{Z})$ is a maximal discrete subgroup of $Sp(6, \mathbf{R})$ [3, Theorem 7], K is equal to $Sp(6, \mathbf{Z})$. □

It follows that for any sequence of elements $g_1, g_2, \ldots$ of $Sp(6, \mathbf{Q})$ which are distinct in the set $Sp(6, \mathbf{Q})/Sp(6, \mathbf{Z})$, the divisors $g_i D$ in the Siegel space are different. Therefore, the ramified double covering of Siegel space pulled back from $M(3) \to X(3)$ gives infinitely many non-isomorphic ramified coverings by the action of $g_1, g_2, \ldots$. Each of these coverings is pulled back from a ramified double covering of some finite level $X(N)$.

For each i, g_i of the Ceresa cycle y in $CH^2(A_{\overline{F}})/l^c$ is nonzero, and $y \neq -y$ (since we showed that $2y \neq 0$). $\mathrm{Gal}(\overline{F}/F)$ acts on that class by the character $\mathrm{Gal}(\overline{F}/F) \to \pm 1$ associated to the translate by g_i of the quadratic extension of F corresponding to $M(3) \to X(3)$. It follows that these infinitely many translates of the Ceresa class are different in $CH^2(A_{\overline{F}})/l^c$. In particular, $CH^2(A_{\overline{F}})/l^c$ is infinite. It follows that $CH^2(A_{\overline{F}})/l$ is infinite. □

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