Rings are understood to be commutative, unless stated otherwise.

(1)(a) Let $R$ be a domain. Show that the polynomial ring $R[x]$ is a domain and that the group of units $R[x]^*$ is equal to $R^*$ (viewed as constant polynomials). Give an example showing that this description of $R[x]^*$ fails for $R$ not a domain, and say where your proof fails. By induction on $n$, it follows that the polynomial ring $A = k[x_1, \ldots , x_n]$ over a field $k$ is a domain, and that $A^* = k^*$.

(1)(b) Show that the power series ring $B = k[[x_1, \ldots , x_n]]$ over a field is also a domain, and find the group of units $B^*$.

(2) Let $A$ and $B$ be commutative rings. The product ring $A \times B$ (not to be confused with a tensor product) is the product set, with ring structure $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$ and $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$. State and prove a universal property that characterizes $A \times B$ in the category of commutative rings. Show that Spec($A \times B$) is the disjoint union of Spec($A$) and Spec($B$), as a set. (In fact, it is the disjoint union as a topological space, but you need not prove that.)

(3) Let $k$ be a field. Since $R = k[x_1, \ldots , x_n]$ is a UFD, the ideal $(f)$ is prime for every irreducible polynomial $f$ in $R$. The following exercise gives some practice in finding irreducible polynomials.

Show that a polynomial in $k[x_1, \ldots , x_n]$ of the form $x_n - f(x_1, \ldots , x_{n-1})$ is irreducible over $k$. Show that a polynomial of the form $x_n^2 - f(x_1, \ldots , x_{n-1})$ is irreducible over $k$ if and only if $f$ is not a square in $k[x_1, \ldots , x_{n-1}]$. 
