Homework 1 for Math 214B Algebraic Geometry

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A variety over a field \( k \) means an integral separated scheme of finite type over \( k \).

Due Monday, Apr. 16.

(1) Let \( k \) be a field. Decompose the scheme \( X = \{(x, y, z) \in A^3_k : x^2 = yz, xz = x\} \) into its irreducible components.

(2) Let \( k \) be a field. Let \( f : A^1_k \to A^1_k \) be an isomorphism of \( k \)-varieties. Prove that \( f \) is given by a polynomial of degree 1.

(3) Let \( k \) be a field. Show that the varieties \( A^1 \) and \( A^1 - \{0\} \) are not isomorphic over \( k \). Likewise for \( A^2 \) and \( A^2 - \{0\} \), which is a bit different.

(4) Let \( X = \{(x, y) \in A^2 : x^2 = y^3\} \) over a field \( k \). Define a bijective morphism from \( A^1 \) to \( X \) over \( k \). Show that this is not an isomorphism. In fact, show that \( A^1 \) and \( X \) are not isomorphic over \( k \).

(5) Let \( k \) be an algebraically closed field of characteristic zero. Find the singular points of the affine curve \( xy + x^3 + y^3 = 0 \) over \( k \).

(6) Show that the conic \( xy = z^2 \) in \( P^2 \) over any field \( k \) is isomorphic to \( P^1 \). Show that every conic (irreducible curve of degree 2 in \( P^2 \)) over an algebraically closed field of characteristic not 2 can be moved to \( xy = z^2 \) by some automorphism of \( P^2 \).

(7) Linear subspaces of \( P^n \). Let \( k \) be a field. A hypersurface in \( P^n \) over \( k \) of degree 1 (that is, defined by a homogeneous polynomial of degree 1) is called a hyperplane. A nonempty intersection of hyperplanes is called a linear subspace of \( P^n \).

(a) If \( Y \) is a linear subspace of dimension \( r \) in \( P^n \), show that \( Y \) is isomorphic to \( P^r \).

(b) Let \( Y, Z \) be linear subspaces of dimension \( r \) and \( s \) in \( P^n \). If \( r + s - n \geq 0 \), then \( Y \cap Z \neq \emptyset \). Moreover, \( Y \cap Z \) is a linear subspace of dimension at least \( r + s - n \) in \( P^n \). (Think of \( A^{n+1}(k) \) as a vector space over \( k \), and work with its subspaces.)

(8) The Veronese embedding. Let \( k \) be a field. For a given \( n, d > 0 \), let \( M_0, \ldots, M_N \) be all the monomials of degree \( d \) in the \( n+1 \) variables \( x_0, \ldots, x_n \), where \( N = \binom{n+d}{d} - 1 \). We define a morphism \( \rho_d : P^n \to P^N \) over \( k \) by sending a
point $P = [a_0, \ldots, a_n]$ to the point $[M_0(a), \ldots, M_N(a)]$ obtained by substituting the numbers $a_i$ in the monomials $M_j$. This is called the $d$th Veronese embedding, or the $d$-uple embedding, of $\mathbb{P}^n$ in $\mathbb{P}^N$. For example, when $n = 1$ and $d = 2$, this is the embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$ as a conic.

(a) Show that the $d$th Veronese map of $\mathbb{P}^1$, $\rho_d : [u, v] \mapsto [u^d, u^{d-1}v, \ldots, v^d]$, is a morphism from $\mathbb{P}^1$ to $\mathbb{P}^d$. Show that the image is closed in $\mathbb{P}^d$. Show that $\rho_d$ is an isomorphism from $\mathbb{P}^1$ to this closed subset, called a rational normal curve in $\mathbb{P}^d$. (These results hold for the Veronese embeddings of $\mathbb{P}^n$ for any $n$, but it takes longer to write out the proofs in general.)

(b) Show that the rational normal curve in $\mathbb{P}^3$ (called the twisted cubic curve) is the projective closure of the affine curve $\{(t, t^2, t^3)\} \subset \mathbb{A}^3$, in some coordinates.