Math 131C Topics in Analysis. Review for Midterm 2

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Review for Midterm 1, Monday, May 16.

The exam will focus on Chapter 2 of the book (Stein-Shakarchi). Of course, the concepts from Chapter 1 will still come up.

Make sure that you know the statements of the main results. Some of the main definitions are short (measurable and integrable functions, $L^1(\mathbf{R}^d)$ and its norm, the Fourier transform on $L^1(\mathbf{R}^d)$ and the Fourier inversion formula); and you should certainly know definitions like that. The definition of the Lebesgue integral (by a 4-step process) is long, and so the exam wouldn't ask you for the whole definition. It's more important to know the main theorems about integration and how to use them.

The Dominated Convergence Theorem is enormously powerful, as I hope you see from the homework problems and from many proofs in the book. The Monotone Convergence Theorem is also useful for dealing with nonnegative functions (before you know that their integrals are finite). The Bounded Convergence Theorem is a special case of DCT, but it's worth being aware of the statement. Finally, Fubini's theorem and (the nonnegative variant) Tonelli's theorem are also powerful, as I hope you see from the results on Fourier transforms and convolutions in the text and the exercises. The lecture only sketched the proof of the Fourier inversion formula in section 2.4, and so you won't be asked about that argument; however, the basic formulas relating the Fourier transform, convolution, derivatives, and so on are very good exercises in using the DCT or Fubini's theorem.

There may be "prove or give a counterexample" questions on the exam. To prepare, try asking yourself variants of the main theorems, as in the sample problems below.

To see the sort of proof expected, make sure you understand all the homework problems in homework sets 3, 4, 5. At least one Exercise (not Problem) will be taken from homework sets 3, 4, or 5, possibly with minor variations.

Some sample questions:

(1) If f_n is an increasing sequence of nonnegative measurable functions on $\mathbf{R}$ with $f_n \to f$ a.e. on $\mathbf{R}$, MCT says that $\int f_n \to \int f \in [0, \infty]$. Is this true for a decreasing sequence of nonnegative measurable functions? (Prove or give a counterexample.) Can you formulate an extra hypothesis that would make this true?

(2) Show that a function f in $L^1(\mathbf{R}^d)$ whose Fourier transform is also in L^1 can be changed on a set of measure 0 to become continuous. (This has a short proof, using basic results about the Fourier transform on L^1.) What's the simplest example f you can give of a function in L^1 that cannot be changed on a set of measure 0 to become continuous? Can you prove that property directly, for your function f?

(3) We showed that for f an integrable function on $\mathbf{R}^d$, and any $\epsilon > 0$, there is

a $\delta > 0$ such that for every measurable set $E \subset \mathbf{R}^d$ of measure less than δ, we have $\int_E |f| < \epsilon$. Can you reprove that result using DCT? (Hint: work out what it would mean for the statement to be false: namely, there would be an $\epsilon > 0$ such that (what?). The argument I am suggesting involves looking at a sequence of functions like $|f|\chi_E$, for varying subsets $E \subset \mathbf{R}^d$. Make sure that you arrange to have all the hypotheses of DCT valid when you apply it.)