Math 131AH, Honors Analysis, UCLA
Sample Midterm 2
Fall 2016

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Midterm 2 will be on Monday, Nov. 14. Below are some sample questions. The exam will cover Rudin's book up to Chapter 3, as in the homeworks up to number 6 (the homeworks handed in so far). At least one quesion will be taken from homeworks 4 to 6, possibly with minor variations.

Make sure you know the definitions and can use them in proofs. An incomplete list of important concepts might include: metric spaces, limit point (of a subset of a metric space), open or closed sets (for example, make sure you understand what an "open subset of [0,1]" means), compact or connected metric spaces, convergent sequences (in a metric space), Cauchy sequences, complete metric spaces, series. Make sure you can give examples that have these properties, or don't have them.

Of course, you still need to know the concepts from the beginning of the course, such as the axioms for the real numbers and the notion of countability.

(1) Define what it means for a series $\sum_{n=1}^{\infty} x_n$ of real numbers to be *convergent*. Determine which of the following series are convergent. Justify your answers.

$$\text{(a)}\ \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right)$$

$$\text{(b)}\ \sum_{n=1}^{\infty} \frac{3^n}{n!}$$

$$\text{(c)}\ \sum_{n=1}^{\infty} \frac{n^2}{n^3+n^2+2}$$

(2) Define what it means for a real number a to be equal to $\limsup_{n\to\infty} s_n$, for a sequence (s_n) of real numbers. For bounded sequences (s_n) and (t_n), show that

$$\limsup_{n\to\infty}(s_n+t_n) \le \limsup_{n\to\infty} s_n + \limsup_{n\to\infty} t_n.$$

Does equality always hold in this formula? Justify your answer.

(3) Let S be a bounded nonempty subset of $\mathbf{R}$ such that $\inf(S)$ is not in S. Show that there is a strictly decreasing sequence of elements of S which converges to $\inf(S)$.

(4) Let X be a metric space. Show that the union of any collection (possibly infinite) of open subsets of X is open in X. Show that the intersection of any finite collection of open subsets of X is open in X. Is the intersection of infinitely many open subsets of X always open in X? Prove or give a counterexample.