Appendix 1, Part 2.

Subjective Probability. We have assumed that the "rational decision maker" knows the exact probabilities of any lottery, $p \in \mathcal{P}^*$, that is offered to him, as he would for example if the lottery were conducted using a roulette wheel. The more general situation also arises in which the probability distribution can only be guessed at, as for example when the payoffs are awarded according to the outcome of a horse race. We do not refer to such situations as a lottery, but rather as a gamble. We assume that our rational person will also have a preference on gambles that satisfies A1 and A2. The question arises: When will the person act as if he had assigned probabilities to the possible outcomes of the horse race and was trying to maximize his expected utility? Such an assignment of probabilities may be considered as that person's *subjective probability* or *personal probability* of the outcomes of the race. This approach to subjective probability is due to Anscombe and Aumann (1963) *Ann. Math. Statist.* **34**, 199-205.

The situation is modelled as follows. We start with a set of payoffs, $\mathcal{P}$, the set $\mathcal{P}^*$ of finite probability distributions on $\mathcal{P}$, and a preference pattern, $\preceq$, on $\mathcal{P}^*$ that satisfies A1 and A2. Let u denote a utility on $\mathcal{P}^*$ that agrees with $\preceq$. Now consider an experiment with finite number, m, of outcomes $\theta_1, \theta_2, \ldots, \theta_m$, one and only one of which is bound to occur. For arbitrary $p_1, p_2, \ldots, p_m$ in $\mathcal{P}^*$, we use $[p_1, p_2, \ldots, p_m]$ to denote the gamble whose payoff is p_j if θ_j occurs, $j = 1, \ldots, m$. Let $\mathcal{G}$ denote the set of all such gambles and let $\mathcal{G}^*$ denote the set of all finite probability distributions on $\mathcal{G}$. An element, $\mathbf{g}$, of $\mathcal{G}^*$ that gives weight to just k elements of $\mathcal{G}$ has the form,

$$\mathbf{g} = (\lambda_1[p_{11}, \ldots, p_{1m}], \ldots, \lambda_k[p_{k1}, \ldots, p_{km}])$$

where the $[p_{i1}, \ldots, p_{im}]$ are elements of $\mathcal{G}$, and $\lambda_1, \ldots, \lambda_k$ are probabilities adding to one.

We assume that there is a preference relation, $\preceq_g$, on $\mathcal{G}^*$ that satisfies A1 and A2. Then by Theorem 1 there exists a utility u_g on $\mathcal{G}^*$ that satisfies the analog of (2), namely

$$\mathbf{g}_1 \preceq \mathbf{g}_2 \qquad \text{if and only if} \qquad u_g(\mathbf{g}_1) \leq u_g(\mathbf{g}_2), \tag{6}$$

We make three assumptions relating the two preference relations, $\preceq$ and $\preceq_g$. Let p_i, p_i', p_{ij}, etc., denote arbitrary elements of $\mathcal{P}$.

A3. *If* $p_i \preceq p_i'$, *then* $[p_1, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_m] \preceq_g [p_1, \ldots, p_{i-1}, p_i', p_{i+1}, \ldots, p_m]$.

A4. *If* $p \prec p'$, *then* $[p, \ldots, p] \prec_g [p', \ldots, p']$.

A5. $(\lambda_1[p_{11}, \ldots, p_{1m}], \ldots, \lambda_k[p_{k1}, \ldots, p_{km}]) \simeq_g [(\lambda_1 p_{11}, \ldots, \lambda_k p_{k1}), \ldots, (\lambda_1 p_{1m}, \ldots, \lambda_k p_{km})]$.

Assumptions A3 and A4 say that the preference relation $\preceq$ on $\mathcal{P}^*$ carries over onto the preference relation $\preceq_g$ on $\mathcal{G}$. Assumption A5 says that every element of $\mathcal{G}^*$ is equivalent$_g$ to the element of $\mathcal{G}$ in which the randomization provided by the λ_j is performed after the outcome of the experiment is observed. As Anscombe and Aumann express it, if the payoff is to be determined by a roulette wheel and a horse race, the decision maker is indifferent whether the roulette wheel is spun before or after the race.

Theorem 2. *If both preference relations* $\preceq$ *on* $\mathcal{P}^*$ *and* $\preceq_g$ *on* $\mathcal{G}^*$ *satisfy A1 and A2, and if Assumptions A3, A4, and A5 are satisfied, then there exist utilities* u *on* $\mathcal{P}^*$ *satisfying (2) and* u_g *on* $\mathcal{G}^*$ *satisfying (6), such that there exist probabilities* $\pi_1, \ldots, \pi_m$ *adding to one with*

$$u_g[p_1, \ldots, p_m] = \pi_1 u(p_1) + \cdots + \pi_m u(p_m). \tag{7}$$

Furthermore, the π_i *are uniquely determined provided there exist* p *and* p' *in* $\mathcal{P}^*$ *such that* $p \prec p'$.

This theorem may be interpreted as follows. If a decision maker has preferences satisfying the conditions of the theorem, then he behaves as if it were known that the probability of outcome θ_i is π_i, $i = 1, \ldots, m$, and his preferences among gambles agrees with preferring gambles with higher expected utility. Thus, π_i can be considered as his subjective probability, or personal probability, of outcome θ_i.

For a proof of Theorem 2, see the paper of Anscombe and Aumann, or Section 1.4 of Ferguson (1968). It might be helpful to see how the π_i are constructed. Let q and q' be any elements of $\mathcal{P}^*$ such that $q \prec q'$. Then from A4, $[q, \ldots, q] \prec_g [q', \ldots, q']$. Since u and u_g are determined only up to change of location and scale, we may choose u so that $u(q) = 0$ and $u(q') = 1$, and u_g so that $u_g[q, \ldots, q] = 0$ and $u_g[q', \ldots, q'] = 1$. Then π_i may be defined as $\pi_i = u_g[q, \ldots, q', \ldots, q]$ with q' in the ith coordinate and q's elsewhere. One then checks that the π_i do not depend on the choice of q and q', and that (7) is satisfied in general.