A GENERAL INVESTMENT MODEL

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Revised May 1985

Abstract. A Bayesian approach to a general investment model is proposed based on knowledge of the probability distributions of possible investment returns (future market events) and of informative dependent variables (future world events). General conditions on these distributions are given under which the simple myopic rule is optimal for log, power and exponential utility functions in finite horizon problems. An example of the model when the myopic rule is not optimal is examined which shows that risk averters may sometimes wager in the face of unfavorable odds.

1. Introduction and Summary. The investment model considered here has two basic components: (1) the sequence of *world events* $\mathbf{Z}_0, \mathbf{Z}_1, \mathbf{Z}_2, \ldots$, assumed to be random variables with values in an arbitrary space, and (2) the sequence of *market events* $\mathbf{Y}_1, \mathbf{Y}_2, \ldots$, assumed to be nonnegative m-dimensional random vectors, $\mathbf{Y}_t = (Y_{t1}, \ldots, Y_{tm})$ for $t = 1, 2, \ldots$. The model is interpreted as occurring sequentially in discrete time periods $t = 1, 2, \ldots$. The events $\mathbf{Z}_0$ represents the history of the world up to the beginning of time period 1, and for $t > 0$ $\mathbf{Z}_t$ represents represents the history of the world during time period t. The positive integer m represents the number of investment opportunities available to the investor, and $Y_{tj} \geq 0$ represents the return per unit invested in the jth investment opportunity during time period t.

At the beginning of time period t, after observing $\mathbf{Z}_0, \mathbf{Y}_1, \mathbf{Z}_1, \ldots, \mathbf{Y}_{t-1}, \mathbf{Z}_{t-1}$, the investor must choose an m-dimensional vector $\mathbf{b}_t = (b_{t1}, \ldots, b_{tm})$ of investments, where b_{tj} represents the amount invested in investment opportunity j during time period t. Let X_0 be a given positive number that represents the investor's initial fortune, and let X_t for $t = 1, 2, \ldots$ denote the investor's fortune at the end of time period t. The investment vectors $\mathbf{b}_t$ are subject to the constraints

$$b_{tj} \geq 0, \quad \text{for } j = 1, \ldots, m \text{ and } t = 1, 2, \ldots \tag{1}$$

and

$$\sum_{j=1}^{m} b_{tj} \leq X_{t-1}, \quad \text{for } t = 1, 2, \ldots. \tag{2}$$

The return from investment $\mathbf{b}_t$ during time period t is $\sum_{j=1}^{m} b_{tj} Y_{tj}$.

Under constraint (1), negative amounts are not allowable investments. Constraint (2) says the investor cannot invest more than he has.

Contained within the information $\mathbf{Z}_{t-1}$ is $R_{t-1} > 0$, the rate at which money expands during period t if placed in the bank. Thus, $R_{t-1} - 1$ is the interest rate for period t. The rate R_{t-1} is known before selecting $\mathbf{b}_t$, even though it might not be known before selecting $\mathbf{b}_{t-1}$. It is assumed that whatever is not invested is placed in the bank. Therefore, the investor's fortune at the end of period t, given investment vector $\mathbf{b}_t$ and fortune X_{t-1} at the beginning of period t, is

$$X_t = R_{t-1}(X_{t-1} - \sum_{j=1}^{m} b_{tj}) + \sum_{j=1}^{m} b_{tj} Y_{tj}. \tag{3}$$

At the end of investment period t, the investor is informed of $\mathbf{Y}_t$ and $\mathbf{Z}_t$, and must choose an investment vector $\mathbf{b}_{t+1}$ for the next period. The investment vector for period t is allowed to depend on all the past world and market events as well as on the investor's initial fortune, and so may be written as

$$\mathbf{b}_t(X_0, \mathbf{Z}_0, \mathbf{Y}_1, \mathbf{Z}_1, \ldots, \mathbf{Y}_{t-1}, \mathbf{Z}_{t-1}), \quad t = 1, 2, \ldots. \tag{4}$$

Note that $\mathbf{b}_t$ depends implicitly on the investor's sequence of fortunes to that time, $X_1, X_2, \ldots, X_{t-1}$. Two basic assumptions are made. It is assumed that the investor views the world as a Bayesian and thus that the joint distribution of $\mathbf{Y}_1, \mathbf{Z}_1, \mathbf{Y}_2, \mathbf{Z}_2, \ldots$ given $\mathbf{Z}_0$ is known to him. Furthermore, it is assumed that the amount he invests in his various opportunities in period t has no influence on the course of future market events $\mathbf{Y}_t, \mathbf{Y}_{t+1}, \ldots$ or on future world events $\mathbf{Z}_t, \mathbf{Z}_{t+1}, \ldots$. More precisely, it is assumed that his distribution of future events is independent of his current and past choices of investments.

We consider the finite horizon problem with horizon n. It is the objective of the investor to maximize the expected value of the utility of his fortune after n time periods. Let $U(x)$ denote the utility to the investor of having fortune $x \geq 0$. Specifically, the investor desires to choose a sequence of investments $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ of the form (4) to maximize

$$\mathrm{E}\{U(X_n) | X_0, \mathbf{Z}_0\} \tag{5}$$

for fixed n subject to constraints (1) and (2), where X_n is defined inductively by (3).

One investment policy attractive for its simplicity is the myopic rule, or one-stage look-ahead rule. This is the rule that for each time period t chooses $\mathbf{b}_t$ to maximize the expected value of the utility of the fortune at the end of the period. That is, $\mathbf{b}_t$ is chosen to maximize

$$\mathrm{E}\{U(X_t) | X_0, \mathbf{Z}_0, \mathbf{Y}_1, \mathbf{Z}_1, \ldots, \mathbf{Y}_{t-1}, \mathbf{Z}_{t-1}\} \tag{6}$$

subject to constraints (1) and (2). From (3), the only random part in the expectation in (6) is $\mathbf{Y}_t$, so that the problem is to choose m numbers $b_{t1}, \ldots, b_{tm}$ subject to linear constraints to maximize an m-dimensional integral. The problem is thus an m-dimensional mathematical programming problem. This is a reasonable problem for machine computation provided m is not too large. Monte Carlo techniques may be used to approximate the m-dimensional integrals provided the distribution of $\mathbf{Y}_t$ given the past is not too bad. The two-stage look-ahead rule, that for each t chooses $\mathbf{b}_t$ as the initial investment of a policy that maximizes the expected utility two stages ahead, is generally better, but besides doubling the the dimensions of the integrals and the decision space, the constraints become nonlinear adding to the complexity. The computation involved in obtaining the optimal policy for the general problem appears intractable.

The objective of this paper is to study conditions on the joint distributions of the world and market events under which the myopic rule is in fact optimal for certain reasonable and well-known utility functions. In Section 2, we treat the power and log utility functions,

$$U_\gamma(x) = \begin{cases} (x^\gamma - 1)/\gamma & \text{for } \gamma \neq 0 \\ \log(x) & \text{for } \gamma = 0. \end{cases} \tag{7}$$

Since utility functions are determined only up to an additive constant and a positive multiplicative constant, we have chosen the $U_\gamma(x)$ so that $U_\gamma(1) = 0$ and $U'_\gamma(1) = 1$ for all γ. In particular, $U_\gamma(x)$ is jointly continuous in γ and x. For $\gamma \leq 0$, $U_\gamma(0)$ is defined to be $-\infty$.

We show that for the utility function $U_0(x)$, the myopic rule is optimal without any further restrictions on the distributions of the world and market events. For the utility functions $U_\gamma(x)$, $\gamma \neq 0$, the myopic rule is optimal under the condition that *for each t, the vectors $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_t, \mathbf{Z}_t$ are conditionally independent given $\mathbf{Z}_0, \mathbf{Z}_1, \ldots, \mathbf{Z}_{t-1}$*. This condition may be broken down into the two conditions: (i) given the world events, the market events are independent, and (ii) given past world events, the future world events are independent of past and present market events. The optimal investment is seen to be proportional to fortune; that is, the optimal $\mathbf{b}_t$ is of the form

$$\mathbf{b}_t(X_0, \mathbf{Z}_0, \mathbf{Y}_1, \ldots, \mathbf{Y}_{t-1}, \mathbf{Z}_{t-1}) = X_{t-1} \cdot \mathbf{c}_t(\mathbf{Z}_0, \mathbf{Y}_1, \ldots, \mathbf{Y}_{t-1}, \mathbf{Z}_{t-1}), \tag{8}$$

where $\mathbf{c}_t$ is a vector of proportions independent of fortune.

This model is an outgrowth of the problem treated by Kelly [15] for the log utility function in which the $(\mathbf{Z}_{t-1}, \mathbf{Y}_t)$ are independent and identically distributed, R_t being identically 1, and the distribution of $\mathbf{Y}_t$ given $\mathbf{Z}_{t-1}$ being multinomial of sample size 1. Bellman and Kalaba, in a series of papers [3-6], extend Kelly's model to independent nonidentically distributed $(\mathbf{Z}_{t-1}, \mathbf{Y}_t)$ and also treat the utility functions U_γ. They also consider exchangeable $\mathbf{Z}_t$, allowing treatment of adaptive problems as in Example 2 below. Additionally, Hakansson [12-14] has examined myopic portfolio policies for these classes of utility functions in a general entrepreneurial context with random market events. The main purpose of this paper is to emphasize the generality under which the main result holds and to point out the difference in the strengths of the assumptions necessary for the different utility functions. A limitation to the model when the myopic rule is not optimal is pointed out at the end of Section 2.

A few examples will aid in understanding the generality of the model.

Example 1. Take one investment opportunity, $m = 1$, $R_t = 1$ for all $t = 0, 1, 2, \ldots$, and $Z_0, Z_1, \ldots$ independent identically distributed uniformly on the interval (0,1). Given the $\mathbf{Z} = (Z_0, Z_1, \ldots)$, take $Y_1, Y_2, \ldots$ independent with $\mathrm{P}(Y_t = 2|\mathbf{Z}) = Z_{t-1}$ and $\mathrm{P}(Y_t = 0|\mathbf{Z}) = 1 - Z_{t-1}$.

The problem may be stated as follows. Every day you are given the opportunity to bet on an even money wager, the probability you win being chosen at random in (0,1) and told to you. How much should you bet each day in order to maximize $\mathrm{E}U_\gamma(X_n)$? This problem satisfies the condition in order that the myopic rule be optimal for all γ.

For logarithmic utility ($\gamma = 0$), the optimal investment is the famous Kelly betting system [15], namely,

$$c_t = \begin{cases} 2Z_{t-1} - 1 & \text{if } Z_{t-1} > 1/2 \\ 0 & \text{if } Z_{t-1} \le 1/2. \end{cases} \tag{9}$$

This betting system has been given a justification independent of the choice of an arbitrary utility function by Breiman [7], by Bell and Cover [2], and by Finkelstein and Whitley [11]. A justification in portfolio selection problems may be found in Thorpe [17,18]. See also Ethier and Tavaré [8].

The general formula for $\gamma < 1$ of the optimal investment proportion is

$$c_t = \begin{cases} \frac{Z_{t-1}^{1/(1-\gamma)} - (1-Z_{t-1})^{1/(1-\gamma)}}{Z_{t-1}^{1/(1-\gamma)} + (1-Z_{t-1})^{1/(1-\gamma)}} & \text{if } Z_{t-1} > 1/2 \\ 0 & \text{if } Z_{t-1} \le 1/2. \end{cases} \tag{10}$$

This illustrates an important rule for risk averters: Never bet on outcomes which have unfavorable odds, or even fair odds.

Example 2. Again take $m = 1$ and $R_t = 1$ for all t. This time take the Z_t independent of the Y_t and so irrelevant for decision making purposes. Let p be unobservable, chosen from a beta distribution with parameters $\alpha > 0$ and $\beta > 0$. Conditionally given p, let $Y_1, Y_2, \ldots, Y_n$ be independent identically distributed with $\mathrm{P}(Y_t = 2|p) = p$ and $\mathrm{P}(Y_t = 0|p) = 1 - p$. This is an adaptive problem in which information as to the value of p gathered as you proceed can influence the amount you choose to invest. Since with p unknown the Y_t are dependent, this example does not satisfy the condition for optimality of the myopic rule when $\gamma \neq 0$.

When $\gamma = 0$, the Kelly betting system with Z_{t-1} replaced by the unbiased estimate of p given the past, namely $(s+\alpha)/(s+f+\alpha+\beta)$, where s is the number of successes and f is the number of failures, is optimal. When $\gamma > 0$, an interesting feature appears. The optimal investment may be positive even though the odds are unfavorable ($\alpha < \beta$). This is discussed at the end of Section 2.

Example 3. For an example of an *adaptive* investment problem that does satisfy the conditions of the optimality of the myopic rule for all γ, modify Example 1 as follows. Take $m = 1$ and $R_t = 1$ for all t. Let θ be unobservable with exponential density $g(\theta) = e^{-\theta}$ for $\theta > 0$, and given θ let $Z_0, Z_1, Z_2, \ldots$ be independent identically distributed with beta density $f(x|\theta) = \theta z^{\theta-1}$ for $0 < z < 1$. Given the Z_t, let the Y_t be independent with $\mathrm{P}(Y_t = 2|\mathbf{Z}) = Z_{t-1}$ and $\mathrm{P}(Y_t = 0|\mathbf{Z}) = 1 - Z_{t-1}$. This problem is adaptive in the sense that learning takes place, that gives information on the size of future payoffs. It is interesting to note though that such learning is irrelevant in the sense that the optimal investment rule is the same as for Example 1. In particular, if $\gamma = 0$, Kelly's betting system is optimal.

In Section 3, a different model is treated in which constraint (2) is removed; the investor is allowed to invest more than he has. Entailed in this generalization is the removal of the requirement that his fortune be nonnegative. This is equivalent to supposing that the investor may borrow unlimited amounts at prevailing interest rates. In this model we are able to treat the utility functions

$$W_\theta(x) = 1 - e^{-\theta x}, \tag{11}$$

defined for all real x, where θ is a given positive number. Under some additional assumptions, the myopic rule is optimal for this model with this utility function, and the optimal investment, $\mathbf{b}_t$, far from being proportional to fortune, turns out to be independent of fortune. This implies that the assumed ability to borrow unlimited amounts, though unrealistic, is of minor importance to a sufficiently wealthy investor. The utility functions, $W_\theta(x)$, have constant absolute risk aversion. In Ferguson [9] they have been seen to arise naturally as the limiting case for a very rich man whose only objective is not to go broke. The constant θ is the logarithm of the rate at which the investor's fortune goes to infinity using an optimal investment policy to minimize the probability of ruin.

2. The Utility functions, U_γ. The objective of this section is to describe conditions under which the myopic rule is optimal for the model (1)-(5), coupled with the utility functions U_γ defined by (7). These are the utility functions, u, that have constant risk aversion, where risk aversion is defined by Arrow [1] and Pratt [16] to be $-xu''(x)/u'(x)$. They are fairly representative, containing the linear utility ($\gamma = 1$) and the logarithmic utility ($\gamma = 0$) as special cases.

The myopic rule for this problem is described in Theorems 1 and 2 in terms of the functions $\mathbf{c}_\gamma(r, F)$ defined as follows. Consider the problem of finding the m-vector $\mathbf{c}$ that maximizes

$$\phi_\gamma(\mathbf{c}) = \mathrm{E}U_\gamma(r + \sum_{i=1}^{m} c_i(Y_i - r)) \tag{12}$$

subject to the constraints

$$c_i \geq 0 \quad \text{for } i = 1, \ldots, m \qquad \text{and} \qquad \sum_{i=1}^{m} c_i \leq 1, \tag{13}$$

where $r > 0$ and the distribution of the random vector $\mathbf{Y} = (Y_1, \ldots, Y_m) \geq 0$ is denoted by F. Let $\mathbf{c}_\gamma(r, F)$ denote the vector $\mathbf{c}$ at which the maximum of (12) subject to (13) is attained.

If $\gamma \geq 1$, the function ϕ_γ is convex on its domain of definition so that it assumes its maximum at one of the extreme points of the constraint set (13). The solution of the problem is then trivial: Find a subscript j such that $\mathrm{E}Y_j^\gamma$ is largest and put $c_j = 1$ unless $\mathrm{E}Y_j^\gamma < r^\gamma$, in which case put all $c_j = 0$. If $\gamma < 1$, the function ϕ_γ is concave over its domain (13), so that the problem of maximizing (12) subject to (13) is a concave programming problem whose solution in principle is straightforward. However, if the number m of investments is large, the evaluation of the m-dimensional integral (12) becomes impractical except in special cases, one of which — the horse race, in which one and only one investment pays off — is discussed in a companion paper [10].

For the log utility function, U_0, no further conditions on the joint distributions of the world and market events are needed to insure the optimality of the myopic rule as described in the following theorem.

Theorem 1. *The optimal rule for the problem of maximizing* $\mathrm{E}\{\log(X_n)|X_0, \mathbf{Z}_0\}$ *out of all rules of the form (4) subject to (1) and (2) is the myopic rule. This rule is the proportional investment rule,*

$$\mathbf{b}_t = X_{t-1}\mathbf{c}_0(R_{t-1}, F_t) \qquad t = 1, \ldots, n$$

where F_t *is the conditional distribution of* $\mathbf{Y}_t$ *given* $\mathbf{Z}_0, \mathbf{Y}_1, \mathbf{Z}_1, \ldots, \mathbf{Y}_{t-1}, \mathbf{Z}_{t-1}$.

Proof. We use the method of backward induction. For the nth period, when the myopic rule is optimal by definition, we seek to choose $\mathbf{b}_n$ to maximize

$$\mathrm{E}\{\log(X_n)|X_0, \mathbf{Z}_0, \ldots, \mathbf{Y}_{n-1}, \mathbf{Z}_{n-1}\} = \\ \mathrm{E}\{\log(R_{n-1}X_{n-1} + \sum_{i=1}^{m} b_{ni}(Y_{ni} - R_{n-1}))|X_0, \mathbf{Z}_0, \ldots, \mathbf{Y}_{n-1}, \mathbf{Z}_{n-1}\}.$$

Write $b_{ni} = X_{n-1}c_{ni}$ for $i = 1, \ldots, m$ and factor out X_{n-1} to obtain

$$\log(X_{n-1}) + \mathrm{E}\{\log(R_{n-1} + \sum_{i=1}^{m} c_{ni}(Y_{ni} - R_{n-1}))|X_0, \mathbf{Z}_0, \ldots, \mathbf{Y}_{n-1}, \mathbf{Z}_{n-1}\} \tag{14}$$

The value of $\mathbf{c}_n$ that maximizes this quantity is clearly $\mathbf{c}_0(R_{n-1}, F_n)$, where F_n is the conditional distribution of $\mathbf{Y}_n$ given $X_0, \mathbf{Z}_0, \ldots, \mathbf{Y}_{n-1}, \mathbf{Z}_{n-1}$, independent of X_0. Since we know we will use this rule at the last stage, the problem at the next to last stage is to choose $\mathbf{b}_{n-1}$ to maximize the expected value of (14) given $X_0, \mathbf{Z}_0, \ldots, \mathbf{Y}_{n-2}, \mathbf{Z}_{n-2}$ subject to (1) and (2). The second term of (14) is independent of $\mathbf{b}_{n-1}$, so this reduces to choosing $\mathbf{b}_{n-1}$ to maximize

$$\mathrm{E}\{\log(X_{n-1})|X_0, \mathbf{Z}_0, \ldots, \mathbf{Y}_{n-2}, \mathbf{Z}_{n-2}\}.$$

Clearly the value of $\mathbf{b}_{n-1}$ that maximizes this is the myopic rule as stated in the theorem, and this process can be continued down to the first stage. $\square$

For $\gamma \neq 0$, the corresponding theorem is not valid without further restrictions placed on the distributions. The exact conditions used in the theorem below are the following.

Condition A1. For $t = 2, \ldots, n$, conditionally given $\mathbf{Z}_0, \mathbf{Z}_1, \ldots, \mathbf{Z}_{t-1}$, the vectors $\mathbf{Y}_t$ and $(\mathbf{Y}_1, \ldots, \mathbf{Y}_{t-1})$ are independent.

Condition A2. For $t = 1, \ldots, n-1$, conditionally given $\mathbf{Z}_0, \mathbf{Z}_1, \ldots, \mathbf{Z}_{t-1}$, the vectors $\mathbf{Z}_t$ and $(\mathbf{Y}_1, \ldots, \mathbf{Y}_t)$ are independent.

These two conditions together are equivalent to the single condition that for $t = 1, \ldots, n$, conditionally given $\mathbf{Z}_0, \mathbf{Z}_1, \ldots, \mathbf{Z}_t$, the vectors $\mathbf{Y}_1, \ldots, \mathbf{Y}_t, \mathbf{Z}_t$ are mutually independent, except that for $t = n$ we do not care about $\mathbf{Z}_n$.

Theorem 2. *For $\gamma < 1$, the myopic rule is optimal for the problem of maximizing* $\mathrm{E}\{U_\gamma(X_n)|X_0,\mathbf{Z}_0\}$ *out of all rules of the form (4) subject to (1) and (2) provided conditions A1 and A2 are satisfied. This rule is the proportional investment rule,*

$$\mathbf{b}_t = X_{t-1}\mathbf{c}_\gamma(R_{t-1},F_t) \qquad t=1,\ldots,n$$

where F_t is the conditional distribution of $\mathbf{Y}_t$ given $\mathbf{Z}_0,\mathbf{Z}_1,\ldots,\mathbf{Z}_{t-1}$.

Proof. For $\gamma = 0$, this is a special case of Theorem 1. Assume $\gamma \neq 0$. For the last period, we seek to choose $\mathbf{b}_n$ to maximize

$$\begin{aligned}
&\mathrm{E}\{X_n^\gamma/\gamma|X_0,\mathbf{Z}_0,\mathbf{Y}_1,\ldots,\mathbf{Y}_{n-1},\mathbf{Z}_{n-1}\}\\
&\qquad = \mathrm{E}\{[R_{n-1}X_{n-1}+\sum b_{ni}(Y_{ni}-R_{n-1})]^\gamma/\gamma|X_0,\mathbf{Z}_0,\mathbf{Y}_1,\ldots,\mathbf{Y}_{n-1},\mathbf{Z}_{n-1}\}\\
&\qquad = X_{n-1}^\gamma\mathrm{E}\{[R_{n-1}+\sum c_{ni}(Y_{ni}-R_{n-1})]^\gamma/\gamma|X_0,\mathbf{Z}_0,\mathbf{Y}_1,\ldots,\mathbf{Y}_{n-1},\mathbf{Z}_{n-1}\}
\end{aligned} \tag{15}$$

where $c_{ni} = b_{ni}/X_{n-1}$. The value of $\mathbf{c}_n$ that maximizes this quantity is $\mathbf{c}_\gamma(R_{n-1},F_n)$ where F_n is the conditional distribution of $\mathbf{Y}_n$ given $\mathbf{Z}_0,\mathbf{Y}_1,\ldots,\mathbf{Y}_{n-1},\mathbf{Z}_{n-1}$, independent of X_0. This distribution and the expectation on the right side of (15) are independent of $\mathbf{Y}_1,\ldots,\mathbf{Y}_{n-1}$ from condition A1. Let the maximum value of (15) be denoted by $(X_{n-1}^\gamma/\gamma)\phi_n(\mathbf{Z}_0,\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1})$, where $\phi_n > 0$. Therefore, the problem at the next to last stage is to choose $\mathbf{b}_{n-1}$ to maximize

$$\begin{aligned}
&\mathrm{E}\{(X_{n-1}^\gamma/\gamma)\phi_n(\mathbf{Z}_0,\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1})|X_0,\mathbf{Z}_0,\mathbf{Y}_1,\ldots,\mathbf{Y}_{n-1},\mathbf{Z}_{n-1}\}\\
&\qquad = X_{n-2}^\gamma\mathrm{E}\{[(R_{n-2}+\sum c_{n-1,i}(Y_{n-1,i}-R_{n-2})^\gamma/\gamma]\phi_n(\mathbf{Z}_0,\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1})\\
&\qquad\qquad |X_0,\mathbf{Z}_0,\mathbf{Y}_1,\ldots,\mathbf{Y}_{n-1},\mathbf{Z}_{n-1}\}
\end{aligned} \tag{16}$$

The latter expectation factors into the product of two expectations from condition A2, and each term separately is independent of $\mathbf{Y}_1,\ldots,\mathbf{Y}_{n-2}$ from conditions A1 and A2, giving (16) in the form

$$\begin{aligned}
&X_{n-2}^\gamma\mathrm{E}\{[R_{n-2}+\sum c_{n-1,i}(Y_{n-1,i}-R_{n-2})]^\gamma/\gamma|\mathbf{Z}_0,\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-2}\}\\
&\qquad\cdot\mathrm{E}\{\phi_n(\mathbf{Z}_0,\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1})|\mathbf{Z}_0,\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-2}\}.
\end{aligned}$$

The value of $\mathbf{c}_{n-1}$ that maximizes this expression is $\mathbf{c}_\gamma(R_{n-2},F_{n-1})$ and the minimum value may be written in the form

$$(X_{n-2}/\gamma)\phi_{n-1}(\mathbf{Z}_0,\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-2}).$$

Clearly this procedure may be continued back to the first stage. $\square$

When conditions A1 and A2 are not both satisfied, the optimal rule is usually difficult to compute. However, in the case of Example 2 with $\gamma \neq 0$, the optimal rule is easy to compute, as was shown by Bellman and Kalaba [5], and the result shows an interesting feature of these models.

Let $f_n(x|\alpha,\beta)$ denote the optimal (maximum if $\gamma > 0$, and minimum if $\gamma < 0$) value of $\mathrm{E}(X_n^\gamma|X_0 = x)$ in Example 2 when the prior distribution of p is $\mathcal{B}e(\alpha,\beta)$, the beta distribution with density proportional to $p^{\alpha-1}(1-p)^{\beta-1}$. The expectation of p is $\alpha/(\alpha+\beta)$, and the posterior distribution of p given a success is $\mathcal{B}e(\alpha+1,\beta)$, and given a failure is $\mathcal{B}e(\alpha,\beta+1)$. Thus, $f_0(x|\alpha,\beta) = x^\gamma$, and

$$\begin{aligned} f_1(x,\alpha,\beta) &= \max_{0\le b\le x}\left\{\frac{\alpha}{\alpha+\beta}(x+b)^\gamma + \frac{\beta}{\alpha+\beta}(x-b)^\gamma\right\} \\ &= \max_{0\le c\le 1} x^\gamma\left\{\frac{\alpha}{\alpha+\beta}(1+c)^\gamma + \frac{\beta}{\alpha+\beta}(1-c)^\gamma\right\} \\ &= x^\gamma V_1(\alpha,\beta), \end{aligned}$$

say, with max replaced by min if $\gamma < 0$. Similarly, we discover $f_k(x|\alpha,\beta) = x^\gamma V_k(\alpha,\beta)$, where V_k is defined recursively by

$$V_k(\alpha,\beta) = \max_{0\le c\le 1}\left\{\frac{\alpha}{\alpha+\beta}V_{k-1}(\alpha+1,\beta)(1+c)^\gamma + \frac{\beta}{\alpha+\beta}V_{k-1}(\alpha,\beta+1)(1-c)^\gamma\right\}, \quad (17)$$

with max replaced by min if $\gamma < 0$, and with initial condition $V_0(\alpha,\beta) = 1$ for all $\alpha > 0$ and $\beta > 0$. Let $c_k(\alpha,\beta)$ represent the value of c at which the max (or min) of (17) is attained. Explicit formulas for $c_k(\alpha,\beta)$ in terms of $V_{k-1}(\alpha+1,\beta)$ and $V_{k-1}(\alpha,\beta+1)$ are easy to obtain by differentiation:

$$c_k(\alpha,\beta) = \left(\frac{1-z}{1+z}\right)^+,$$

where

$$z = \left(\frac{\beta V_{k-1}(\alpha,\beta+1)}{\alpha V_{k-1}(\alpha+1,\beta)}\right)^{1/(1-\gamma)}, \quad \gamma < 1.$$

Of course, it might be expected that $c_k(\alpha,\beta) = 0$ whenever $\alpha < \beta$. After all, the investor then does not believe the investment to be favorable and he has convex utility. According to Arrow [1], a risk averter is one who is unwilling to make fair or subfair bets. In addition, $c_k(\alpha,\beta)$ does equal zero when ever $\alpha \le \beta$ for $\gamma = 0$ since the optimal investment policy is then the Kelly betting system. However, for $\gamma > 0$, $\alpha \le \beta$ does not imply that $c_k(\alpha,\beta) = 0$ as Table 1 constructed for the value $\gamma = 1/2$ illustrates.

For example, if $n = 14$ and the prior is uniform ($\alpha = \beta = 1$), the optimal investment is $c_{14}(1,1) = 97.3$ *percent* of the investor's fortune; furthermore, if the investor should lose that investment and then believe the odds are two to one against him, he should still invest $c_{13}(1,2) = 11.7$ *percent* of his remaining fortune!

This brings out clearly one aspect of this model that occurs when the myopic rule is not optimal. In various other multistage decision problems such as statistical sequential

Table 1. Optimal values of $c_n(\alpha, \beta)$, $\gamma = 1/2$.

n	$c(1,1)$	$c(1,2)$	$c(2,1)$	$c(1,3)$	$c(2,2)$	$c(3,1)$
1	0.000	0.000	0.600	0.000	0.000	0.800
2	0.053	0.000	0.667	0.000	0.020	0.843
3	0.143	0.000	0.734	0.000	0.057	0.897
4	0.257	0.000	0.795	0.000	0.107	0.908
5	0.385	0.000	0.847	0.000	0.170	0.930
6	0.516	0.000	0.888	0.000	0.245	0.947
7	0.636	0.000	0.919	0.000	0.331	0.959
8	0.738	0.000	0.941	0.000	0.423	0.969
9	0.817	0.000	0.957	0.000	0.518	0.975
10	0.875	0.000	0.968	0.000	0.608	0.980
11	0.915	0.000	0.975	0.000	0.691	0.984
12	0.943	0.000	0.981	0.000	0.762	0.986
13	0.961	0.117	0.985	0.000	0.819	0.988
14	0.973	0.278	0.987	0.000	0.864	0.990
15	0.980	0.434	0.989	0.000	0.898	0.991
16	0.985	0.573	0.991	0.000	0.923	0.992
17	0.989	0.688	0.992	0.000	0.941	0.993
18	0.991	0.777	0.993	0.000	0.954	0.994
19	0.992	0.842	0.994	0.000	0.963	0.995
20	0.993	0.889	0.995	0.000	0.969	0.995
21	0.994	0.921	0.995	0.000	0.974	0.996
22	0.995	0.944	0.996	0.065	0.978	0.996
23	0.995	0.959	0.996	0.241	0.980	0.996
24	0.996	0.969	0.996	0.409	0.983	0.997
25	0.996	0.976	0.997	0.555	0.984	0.997

analysis or Markovian decision problems, one solves the problem for a large finite horizon letting the horizon go to infinity in order to get an idea of what a good decision rule is for the immediate next stage. However, in this example, it has been shown by Chia-Jon Hong that $c_n(\alpha, \beta) \to 1$ as $n \to \infty$. No matter how unfavorable the initial prior, for a sufficiently large horizon one would invest an arbitrarily high percentage of one's fortune at the first stage.

In some related problems, a similar phenomenon occurs because by investing we gain information that is useful in future investments. This is not occuring here since we get the information about investments whether or not we invest. Here, the explanation is that if the true p is close to 1, we want to be in a good position to take advantage of that fact, because then the return will be exponentially large in n, and if $0 < \gamma < 1$ the utility of the return will also be exponentially large in n.

The corresponding phenomenon for $\gamma < 0$ is similar. Here one does avert risk in that $c_n(\alpha, \beta) = 0$ if $\alpha \leq \beta$. However, it seems from Table 2, constructed for $\gamma = -1$, that $c_n(\alpha, \beta) \to 0$ as $n \to \infty$ for all a and β. No matter how favorable the initial odds, for a

Table 2. Optimal values of $c_n(\alpha, \beta)$, for $\gamma = -1$.

n	$c(2,1)$	$c(3,1)$	$c(3,2)$	$c(4,1)$	$c(4,2)$	$c(5,1)$
1	0.172	0.268	0.101	0.333	0.172	0.382
2	0.155	0.246	0.094	0.309	0.161	0.358
3	0.139	0.226	0.087	0.287	0.151	0.335
4	0.124	0.208	0.080	0.268	0.142	0.314
5	0.111	0.192	0.074	0.250	0.134	0.295
6	0.098	0.177	0.067	0.233	0.126	0.278
7	0.087	0.164	0.062	0.218	0.119	0.262
8	0.077	0.152	0.056	0.205	0.112	0.247
9	0.068	0.141	0.051	0.193	0.106	0.234
10	0.060	0.131	0.047	0.182	0.100	0.222
11	0.052	0.122	0.042	0.171	0.095	0.210
12	0.045	0.114	0.038	0.162	0.090	0.200
13	0.039	0.107	0.034	0.153	0.085	0.191
14	0.033	0.100	0.031	0.146	0.081	0.182
15	0.028	0.094	0.027	0.138	0.077	0.174
16	0.023	0.088	0.024	0.132	0.073	0.167
17	0.018	0.082	0.021	0.126	0.069	0.160
18	0.014	0.077	0.018	0.120	0.066	0.153
19	0.010	0.073	0.015	0.115	0.063	0.147
20	0.060	0.068	0.013	0.110	0.059	0.142
21	0.003	0.064	0.010	0.105	0.057	0.136
22	0.000	0.060	0.008	0.101	0.054	0.132
23	0.000	0.057	0.006	0.096	0.051	0.127
24	0.000	0.053	0.003	0.093	0.049	0.123
25	0.000	0.050	0.001	0.089	0.046	0.119

sufficiently large horizon one would invest an arbitrarily small percentage of one's fortune at the first stage. Again, this model is not designed for problems for deciding what to do now based on what is good for sufficiently large horizons. We conjecture that for all α, β and $\gamma < 0$, the optimal betting proportion $c_n(\alpha, \beta)$ is equal to zero for sufficiently large n.

3. The Utility Functions, W_θ. In this section, the basic model is altered to allow arbitrary real values for fortunes, $-\infty < x < \infty$, and to allow the investor to borrow unlimited amounts of money at prevailing rates. We assume the investor has a utility function $W_\theta(x)$ of the form (11) for a given positive constant $\theta > 0$. The utility $W_\theta(x)$, defined for all real values of x, is concave and bounded above. This is the class of utility functions with constant absolute risk aversion, where absolute risk aversion is defined by Arrow and Pratt to be $-u'(x)/u''(x)$.

Under two assumptions in addition to A1 and A2, the myopic rule is optimal for the problem of maximizing $\mathrm{E}\{W_\theta(X_n)|X_0, \mathbf{Z}_0\}$. The first of these assumptions is that the sequence of growth factors, $R_0, R_1, \ldots, R_n$ be identically one.

Condition A3. $R_0 = R_1 = \cdots = R_{n-1} = 1$ a.s.

The second of these additional assumptions is more technical. Since we are allowing unlimited borrowing we need a condition to avoid the degenerate case of desiring to borrow an infinite amount. If there is an investment policy that provides a return greater than one with probability one, then the investor could make his wealth large without bound by investing a sufficiently large amount in this policy. Condition A4 rules out this possibility.

Condition A4. For every $t = 1, \ldots, n$ and every m-vector $\mathbf{b} \geq 0$ with $\mathbf{b} \neq \mathbf{0}$,

$$\mathrm{P}\{\sum_1^m b_i(Y_{ti} - 1) < 0 | \mathbf{Z}_0\} > 0.$$

Consider the problem of finding an m-vector $\mathbf{b}$ that maximizes $\mathrm{E}W_\theta(x + \sum_1^m b_i(Y_i - 1))$ subject to the constraints $b_i \geq 0$ for $i = 1, \ldots, m$. It is assumed that x, the investor's fortune, is a known constant and only the vector $\mathbf{Y} = (Y_1, \ldots, Y_m)$ is random. This problem is equivalent to the one of finding $\mathbf{b}$ to *minimize*

$$\phi(\mathbf{b}) = \mathrm{E}\exp\{-\theta \sum_1^m b_i(Y_i - 1)\}, \tag{18}$$

subject to the constraints $b_i \geq 0$ for $i = 1, \ldots, m$. The optimal $\mathbf{b}$ is thus seen to be independent of x. If, as we assume, $\mathrm{P}(\sum_1^m b_i(Y_i - 1) < 0) > 0$ for all $\mathbf{b} \geq 0$, $\mathbf{b} \neq 0$, then the minimum of (18) over $\mathbf{b} \geq 0$ is attained at some finite $\mathbf{b}$. Let F denote the distribution function of $\mathbf{Y}$, and let $\mathbf{b}_\theta(F)$ denote any vector $\mathbf{b}$ that attains the minimum of (18) subject to $\mathbf{b} \geq 0$. Define $\mathbf{b}^*(F)$ to be $\mathbf{b}_1(F)$; then we may write $\mathbf{b}_\theta(F) = \mathbf{b}^*(F)/\theta$.

Theorem 3. *Under conditions A1, A2, A3 and A4, the myopic rule*

$$\mathbf{b}_t = \mathbf{b}^*(F_t)/\theta, \qquad \textit{for } t = 1, 2, \ldots, n$$

is optimal for maximizing $\mathrm{E}\{W_\theta(X_n) | X_0, \mathbf{Z}_0\}$ *out of all rules of the form (4) subject to (1), where* F_t *is the conditional distribution of* $\mathbf{Y}_t$ *given* $\mathbf{Z}_0, \mathbf{Z}_1, \ldots, \mathbf{Z}_{t-1}$.

The proof is similar to the proof of Theorem 2 and is omitted.

The optimal rule can be found just as easily if A3 is replaced by the condition that the growth factors are deterministic, that is the R_t are positive constants known at time zero. The myopic rule is not optimal in this case but a closely related rule is. This is the rule that at each stage t minimizes

$$\mathrm{E}\{\exp\{-\theta(\prod_{j=t}^{n-1} R_j) \sum_{i=1}^m b_i(Y_{ti} - R_{t-1})\} | \mathbf{Z}_0, \mathbf{Z}_1, \ldots, \mathbf{Z}_{t-1}\}.$$

In terms of the function $\mathbf{b}^*$, this rule is

$$\mathbf{b}_t = \mathbf{b}^*(F_t)/(\theta \prod_{j=t}^{n-1} R_j), \qquad \text{for } t = 1, 2, \ldots, n, \tag{19}$$

Table 3. Optimal values of $b_n(\alpha, \beta)$ for $\theta = 1$.

n	$b(2,1)$	$b(3,1)$	$b(3,2)$	$b(4,1)$	$b(4,2)$	$b(5,1)$
1	0.347	0.549	0.203	0.693	0.347	0.805
2	0.275	0.448	0.173	0.576	0.307	0.677
3	0.219	0.374	0.148	0.489	0.265	0.582
4	0.175	0.318	0.126	0.423	0.235	0.509
5	0.139	0.275	0.107	0.372	0.211	0.452
6	0.110	0.239	0.090	0.331	0.189	0.405
7	0.086	0.211	0.076	0.297	0.171	0.368
8	0.066	0.187	0.063	0.269	0.155	0.336
9	0.048	0.166	0.051	0.246	0.141	0.309
10	0.033	0.148	0.041	0.225	0.129	0.286
11	0.020	0.133	0.031	0.208	0.118	0.266
12	0.008	0.119	0.023	0.192	0.108	0.249
13	0.000	0.107	0.015	0.178	0.099	0.233
14	0.000	0.097	0.008	0.166	0.090	0.219
15	0.000	0.087	0.001	0.155	0.083	0.207
16	0.000	0.078	0.000	0.145	0.076	0.195
17	0.000	0.070	0.000	0.135	0.069	0.185
18	0.000	0.062	0.000	0.127	0.063	0.176
19	0.000	0.055	0.000	0.119	0.057	0.167
20	0.000	0.049	0.000	0.112	0.052	0.159
21	0.000	0.043	0.000	0.105	0.047	0.151
22	0.000	0.037	0.000	0.099	0.042	0.144
23	0.000	0.032	0.000	0.093	0.038	0.138
24	0.000	0.027	0.000	0.088	0.034	0.132
25	0.000	0.022	0.000	0.083	0.030	0.126

where this time F_t represents the distribution function of $\mathbf{Y}_t/R_{t-1}$ given $\mathbf{Z}_0\mathbf{Z}_1, \ldots, \mathbf{Z}_{t-1}$. We see that, like the problems of Section 2 in which the myopic rule was not optimal, one cannot use the initial rule of an optimal policy for a large horizon as a sensible initial rule for an infinite horizon. In particular, if $\prod_0^n R_t \to \infty$ as $n \to \infty$, then we expect $\mathbf{b}_1 \to 0$.

When the myopic rule is not optimal for reasons such as those found in Example 2, remarks similar to those of Section 2 apply. If in Example 2, $b_n(\alpha, \beta)$ denotes the amount of the optimal investment when the prior distribution is $\mathcal{B}e(\alpha, \beta)$ and there are n stages to go, then $b_n(\alpha, \beta) = 0$ for $a \leq \beta$. Table 3 contains results for Example 2 in the case $\theta = 1$ and indicates that $b_n(\alpha, \beta) = 0$ for sufficiently large n.

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