

LAST ROUND BETTING

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Abstract

Two players with differing amounts of money simultaneously choose an amount to bet on an even-money win-or-lose bet. The outcomes of the bets may be dependent and the player who has the larger amount of money after the outcomes are decided is the winner. This game is completely analyzed. In nearly all cases, the value exists and optimal strategies for the two players that give weight to a finite number of bets are explicitly exhibited. In a few situations, the value does not exist.

ZERO-SUM GAME, GENERALIZED SADDLE POINT, JEOPARDY

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0. Introduction

This paper contains a mathematical analysis of a two-person zero-sum game that models last round betting situations occurring in certain gambling tournaments and in the television game of Jeopardy. The game of Final Jeopardy has been analyzed in fairly realistic detail in papers of Taylor (1994) and Gilbert and Hatcher (1994). Taylor's analysis covers three players and his suggested strategies are intended to work well against the historical choices of the players in past games. The paper of Gilbert and Hatcher contains a mathematical analysis of Final Jeopardy as a two or three person nonzero-sum game

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(evidently a tie is almost as good as a win), and the emphasis is on finding equilibrium points. The book of Wong (1992) contains much useful information about playing casino tournaments in general. In most of these tournaments, the bets are made sequentially by the players according to a predetermined order, whereas our interest is in games with simultaneous moves. However, in some blackjack and baccarat tournaments each player is allowed to keep the amount bet a secret at one time chosen by the player. If the advice of Wong is followed, this time is chosen to be in the last round.

It is this situation of two players making last round bets with simultaneous moves that is treated in this paper. It is not uncommon that the last round of a tournament reduces essentially to a battle between the two leaders. In addition, the game-theoretic solution of the two-person game sheds light on the problems arising when three or more players still have a chance to win. These games are similar to the Colonel Blotto games with continuously divisible resources and 2 outposts. In Colonel Blotto games, two players must decide how to divide their resources between n outposts of differing values, the player with the larger resource at an outpost winning the value of the outpost. The papers of Gross and Wagner (1950) and Gross (1950) contain solutions to the Colonel Blotto games for $n = 2$ and arbitrary resources, and for all n when the players have equal resources. In all cases studied, the games have values, but it is unknown whether these games always have values. For a variation of Colonel Blotto without a value, see Sion and Wolfe (1957). Last round betting games are in spirit similar to games of timing, discussed in Karlin (1959). Such games are played on the unit square with a payoff function that is discontinuous on the main diagonal. There is a large literature on such games that continues today. See for example Kimeldorf and Lang (1978), Radzik (1988), Teraoka and Nakai (1990), and Kurisu (1992). Also similar are the Silverman games as treated for example in Heuer (1989), and certain exchange games such as the MIX game of Garnaev (1992).

1. The Rules

The last round is modeled as follows. The game begins with Player I having an amount $X > 0$ of money and Player II having an amount $Y > 0$. We shall assume money is infinitely divisible and take $X \leq Y = 1$ without loss of generality. Player I chooses an

amount x to bet, where $0 \leq x \leq X$, and simultaneously, II chooses an amount y where $0 \leq y \leq 1$. The players are then tested and the players separately either win or lose the amount they bet. There may be a dependence in the outcome so we let p_{11} denote the probability that they both win, p_{00} denote the probability they both lose, p_{10} denote the probability that I wins and II loses, and p_{01} denote the probability that I loses and II wins. We have $p_{00} + p_{10} + p_{01} + p_{11} = 1$. In both Jeopardy and blackjack, there is a positive dependence in the outcomes of the two players.

The player who has the larger amount of money after the play wins the game. If the amounts are equal, then as we shall see, except for a few values of X , the actual definition of the winner is immaterial. To keep things simple, we call the game a tie when the final amounts are equal, or equivalently, we assume the winner is decided by the toss of a fair coin. Then the probability that player I wins when I uses x and II uses y is

$$\begin{aligned} P(x, y) &= p_{11}I(X + x > 1 + y) + p_{10}I(X + x > 1 - y) + p_{00}I(X - x > 1 - y) \\ &\quad + \frac{1}{2}[p_{11}I(X + x = 1 + y) + p_{10}I(X + x = 1 - y) \\ &\quad + p_{00}I(X - x = 1 - y) + p_{01}I(X - x = 1 + y)] \end{aligned}$$

where $I(A)$ denotes the indicator function of the set A : $I(A) = 1$ if the set A obtains, and $I(A) = 0$ otherwise. See Figure 1. The pure strategy spaces of players I and II are denoted by $\mathcal{X} = [0, X]$ and $\mathcal{Y} = [0, 1]$ respectively, and the game is denoted by $G = (\mathcal{X}, \mathcal{Y}, P)$.

Since the payoff function, $P(x, y)$, is not upper or lower semi-continuous, we are not assured of the existence of a value for these games. Nevertheless, except for a discrete set of values of X , the value of the game exists and there are optimal mixed strategies for the players giving probability one to a finite set of points. After treating the special cases $p_{11} = 0$, $p_{00} = 0$, $X < 2/3$ and $X = 1$ in Section 2, the general case is treated in Section 3. The solution is seen to depend upon which interval of the form $[\frac{k}{k+1}, \frac{k+1}{k+2})$, for integer k , contains X . Theorem 1 treats the case of k even, and Theorems 2 and 3 the case of k odd. The value and finite optimal strategies exist in all cases except when $0 < p_{11} < p_{10}$ and X is equal to one of $3/4, 5/6, \dots$. In these exceptional cases, the lower and upper values are found and seen to be distinct. When the value exists, optimal strategies are found that

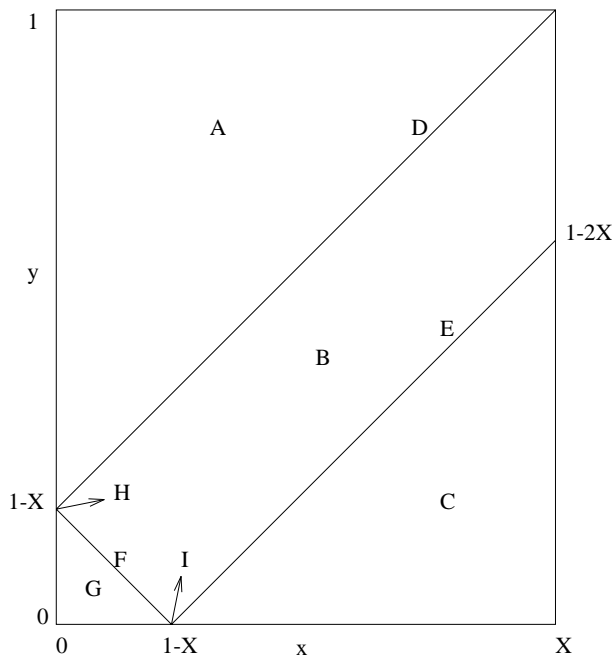


Figure 1. The Payoff Function, P . I chooses x , II chooses y . The Payoffs A, B, \dots, I for the indicated regions of $\mathcal{X} \times \mathcal{Y}$ are given by $A = p_{10} + p_{00}$, $B = p_{10}$, $C = p_{10} + p_{11}$, $D = p_{10} + (1/2)p_{00}$, $E = p_{10} + (1/2)p_{11}$, $F = (1/2)p_{10}$, $G = 0$, $H = (1/2)(p_{10} + p_{00})$, $I = (1/2)(p_{10} + p_{11})$. The inequalities $A \geq D \geq B$, $C \geq E \geq B$, $G \leq F \leq B$, and $F \leq \max(H, I)$ are helpful towards establishing dominance relationships for the strategies of the two players.

give weights in geometric proportion to a set of equally spaced points, except for at most one endpoint.

2. Saddle Point Cases

We first treat a few special cases in which there is a saddle point. This will allow us to assume $2/3 \leq X < 1$, $p_{00} > 0$ and $p_{11} > 0$ in the following section. Case 1 is treated explicitly and the rest of the cases may be shown in a similar manner.

Case 1: $X < 1/2$. Figure 1 simplifies in this case, and the value and optimal strategies are immediate. Indeed, for (a) $1 - X < y \leq 1 \Rightarrow \max_x P(x, y) = p_{10} + p_{00}$ for $x < X + y - 1$, (b) $y = 1 - X \Rightarrow \max_x P(x, y) = (1/2)p_{10} + (1/2) \max(p_{10}, p_{00})$ for $x > 0$ if $p_{00} < p_{10}$ and $x = 0$ if $p_{00} \geq p_{10}$, (c) $1 - 2X < y < 1 - X \Rightarrow \max_x P(x, y) = p_{10}$ for $x > 1 - y - X$, (d) $y = 1 - 2X \Rightarrow \max_x P(x, y) = (1/2)p_{10}$ for $x = X$, (e) $0 \leq y < 1 - 2X \Rightarrow \max_x P(x, y) = 0$

for $0 \leq x \leq X$. The discussion implies that Player II can secure 0 by betting anything less than $1 - 2X$, i.e. that $\min_y \max_x P(x, y) = 0$. On the other hand, $0 \leq x \leq X \Rightarrow \min_y P(x, y) = 0$ for $y < 1 - 2X$, i.e. $\max_x \min_y P(x, y) = 0$ for $0 \leq x \leq X$. Hence, this is the case where Player II always wins by betting nothing.

The value is 0.

Anything is optimal for player I.

An optimal strategy for player II is $y = 0$.

Case 2: $X = 1/2$. To create a chance of winning, player I must bet all he has. In this case the tie rule plays a role in the solution.

The value is $\min\{p_{10}, (p_{10} + p_{11})/2\}$.

Player I's optimal strategy is $x = 1/2$.

Player II's optimal strategy is $y = 0$ if $p_{11} \leq p_{10}$

and any y in $(0, 1/2)$ if $p_{11} > p_{10}$.

Case 3: $1/2 < X < 2/3$. Player I may as well bet all he has.

The value is p_{10} .

Any x in $(1 - X, X]$ is optimal for Player I.

Any y in $(2X - 1, 1 - X)$ is optimal for Player II.

Case 4: $p_{11} = 0$. Player II cannot gain by betting.

$(X, 0)$ is a saddle point.

If $1/2 < X < 1$, the value is p_{10} .

Case 5: $p_{00} = 0$. Player I may as well bet all he has.

$(X, 1)$ is a saddle point.

If $1/2 < X < 1$, the value is p_{10} .

Case 6: $X = 1$. In this case, there is always a saddle point occurring at one of the corner points of the product strategy space, which corner depends on the values of the p_{ij} .

If $p_{11} \geq p_{00}$, then $(1, 1)$ is a saddle point with value $p_{10} + (p_{11} + p_{00})/2$.

If $1/2 \leq p_{10} + p_{11} \leq p_{10} + p_{00}$, then $(1, 0)$ is a saddle point with value $p_{10} + p_{11}$.

If $p_{10} + p_{11} \leq 1/2 \leq p_{10} + p_{00}$, then $(0,0)$ is a saddle point with value $1/2$.

If $p_{10} + p_{11} \leq p_{10} + p_{00} \leq 1/2$, then $(0,1)$ is a saddle point with value $p_{10} + p_{00}$.

3. The General Case

Throughout this section, it is assumed that $2/3 \leq X < 1$, $p_{00} > 0$ and $p_{11} > 0$. It turns out that none of these games has a saddle point, so we must consider the mixed extension of $G(\mathcal{X}, \mathcal{Y}, P)$. It is simplest to take the set of finite probability distributions over \mathcal{X} and \mathcal{Y} as the classes of mixed strategies of the players. For a set \mathcal{X} , we let \mathcal{X}^* denote the class of finite probability distributions over \mathcal{X} . We may consider \mathcal{X} as a subset of \mathcal{X}^* by identifying a point $x \in \mathcal{X}$ with the probability distribution in \mathcal{X}^* giving mass 1 to the point x .

Suppose $\sigma \in \mathcal{X}^*$ is the probability distribution giving probability σ_i to the point x_i for $i = 1, \dots, m$ and suppose $\tau \in \mathcal{Y}^*$ gives mass τ_j to the point y_j for $j = 1, \dots, n$. In the usual way, we may then extend the definition of P from $\mathcal{X} \times \mathcal{Y}$ to all of $\mathcal{X}^* \times \mathcal{Y}^*$ by

$$P(\sigma, \tau) = \sum_{i=1}^m \sum_{j=1}^n \sigma_i \tau_j P(x_i, y_j).$$

The mixed extension of $G = (\mathcal{X}, \mathcal{Y}, P)$ is then $G^* = (\mathcal{X}^*, \mathcal{Y}^*, P)$.

The restriction to finite probability distributions over \mathcal{X} and \mathcal{Y} is also the strongest in the following sense. Let \mathcal{X}^{**} (resp. \mathcal{Y}^{**}) represent any convex family of distributions over \mathcal{X} (resp. \mathcal{Y}) containing all distributions degenerate at points. Then $\mathcal{X}^* \subset \mathcal{X}^{**}$ and $\mathcal{Y}^* \subset \mathcal{Y}^{**}$. If the value exists for the mixed extension $(\mathcal{X}^*, \mathcal{Y}^*, P)$, then it also exists for the mixed extension $(\mathcal{X}^{**}, \mathcal{Y}^{**}, P)$ and the values are equal. (If P is bounded as it is here, the converse also holds. However if the probability measures are not required to be σ -additive, an example of Baston, Bostock and Ruckle (1990) shows the converse might not hold even if P is bounded.)

All these games may be solved using the method of block domination to reduce the strategy spaces, \mathcal{X} and \mathcal{Y} , to finite sets. For a given game, $G = (\mathcal{X}, \mathcal{Y}, P)$, a pair of subsets, $(\mathcal{X}_0, \mathcal{Y}_0)$ with $\mathcal{X}_0 \subseteq \mathcal{X}$ and $\mathcal{Y}_0 \subseteq \mathcal{Y}$, is said to be a **generalized saddle-point** for G if

- (a) for every $x \in \mathcal{X}$, there is a $\sigma \in \mathcal{X}_0^*$ such that $P(\sigma, y) \geq P(x, y)$ for all $y \in \mathcal{Y}_0$, and
- (b) for every $y \in \mathcal{Y}$, there is a $\tau \in \mathcal{Y}_0^*$ such that $P(x, \tau) \leq P(x, y)$ for all $x \in \mathcal{X}_0$.

The conditions (a) and (b) may be stated in more intuitive terms:

- (a) if II is restricted to \mathcal{Y}_0 , then I may restrict his attention to \mathcal{X}_0 , and
- (b) if I is restricted to \mathcal{X}_0 , then II may restrict his attention to \mathcal{Y}_0 .

The term generalized saddle-point is found in Shapley (1959). The following lemma is well-known (see for example, Karlin (1959, vol I, Theorem 2.2.4).

Lemma 1. *If $(\mathcal{X}_0, \mathcal{Y}_0)$ is a generalized saddle point for the game $(\mathcal{X}, \mathcal{Y}, P)$, then the restricted game $(\mathcal{X}_0, \mathcal{Y}_0, P)$ has a value if and only if the original game $(\mathcal{X}, \mathcal{Y}, P)$ has a value, the values are equal, and any strategy optimal in the restricted game (for either player) is also optimal in the original game.*

We now define $\Delta := 1 - X$ and consider three cases.

Case 1: The left-closed intervals, $[\frac{2}{3}, \frac{3}{4}), [\frac{4}{5}, \frac{5}{6}), \dots$

$$X \in \left[\frac{2m}{2m+1}, \frac{2m+1}{2m+2} \right), \quad m = 1, 2, \dots$$

Take $\epsilon > 0$ sufficiently small ($\epsilon \leq \frac{2(m+1)\Delta-1}{m}$ will do) and define subsets $\mathcal{X}_{1,m} = \{x_\rho, \rho = 0, 1, \dots, m\} \subset \mathcal{X}$ and $\mathcal{Y}_{1,m} = \{y_\rho, \rho = 0, 1, \dots, m\} \subset \mathcal{Y}$ where

$$\begin{aligned} x_0 &= 0, & x_\rho &= 1 - (2m+1 - 2\rho)\Delta & \rho &= 1, \dots, m \\ y_0 &= 0, & y_\rho &= 1 - (2m+2 - 2\rho)\Delta + (m - \rho + 1)\epsilon & \rho &= 1, \dots, m. \end{aligned}$$

See Figure 2.

Lemma 2. *For Case 1, if Player II is restricted to $\mathcal{Y}_{1,m}$, then Player I may restrict his attention to $\mathcal{X}_{1,m}$.*

Proof: Let us assume that II is restricted to $\mathcal{Y}_{1,m}$. Then, (a) For all $x \in [0, \Delta)$, $X \in \left[\frac{2m}{2m+1}, \frac{2m+1}{2m+2} \right)$, $m = 1, 2, \dots$ implies $y_1 > \Delta$ and therefore, $P(x, y_1) \leq P(x_0, y_1)$, and $P(x, y_j) = P(x_0, y_j)$, $j = 0, 2, \dots, m$ (see Figures 1 and 2). Hence, x_0 dominates all $x \in [0, \Delta)$. (b) For all $x \in [\Delta, x_1 + m\epsilon)$, (i) $P(x, y_0) \leq P(x_1, y_0)$ (actually, $P(x, y_0)$ is constant for $x \in (\Delta, x_1 + m\epsilon)$), (ii) The upper inequality on ϵ implies $y_1 < 2\Delta$ and therefore $P(x, y_1) = P(x_1, y_1) = p_{10}$, (iii) $y_2 < x_1 + m\epsilon + \Delta$ implies $P(x, y_2) \leq P(x_1, y_2)$, and finally,

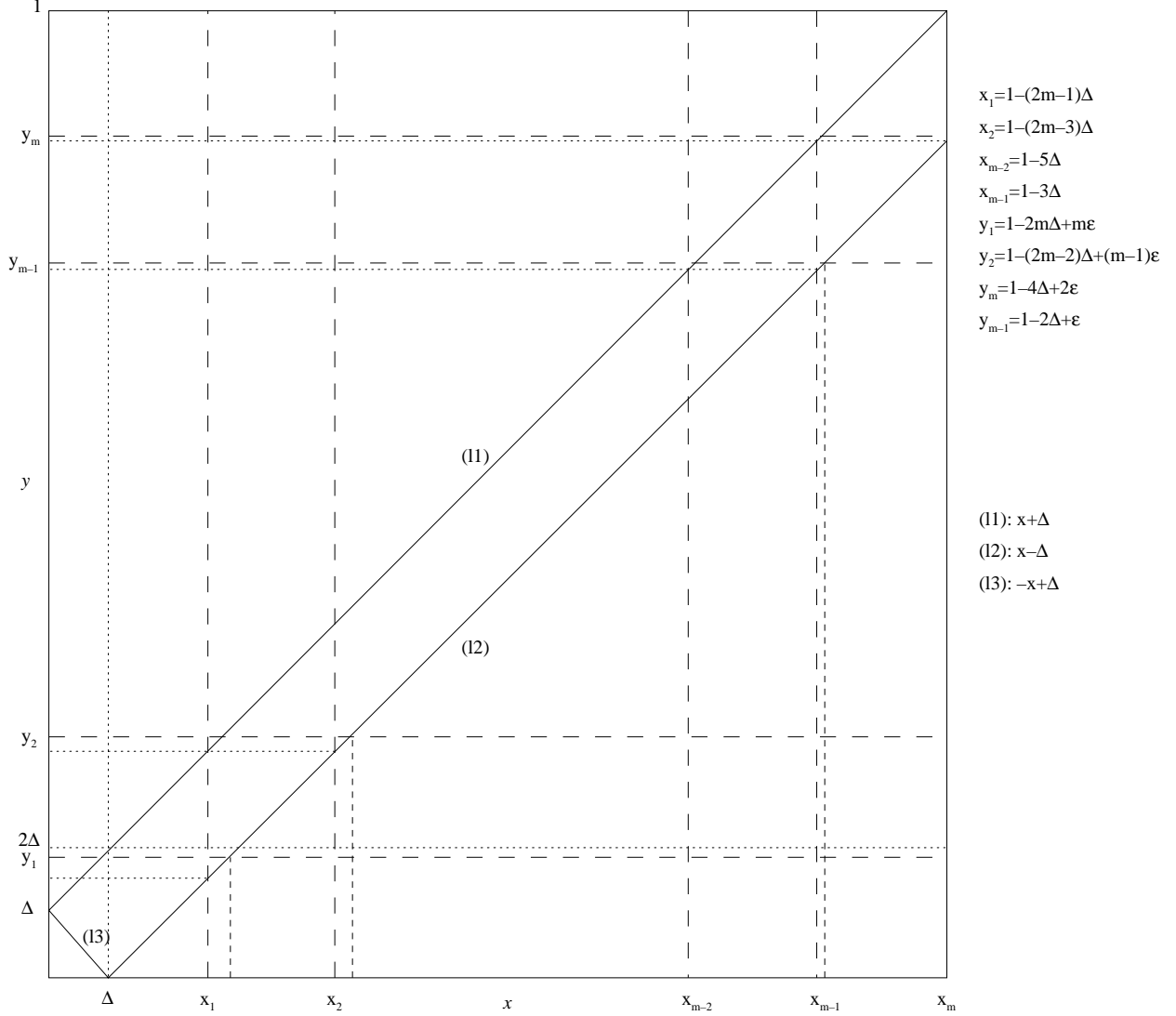


Figure 2. Case 1. The active pure strategies of I and II are represented by the widely spaced dashed lines.

(iv) $P(x, y_j)$ is constant for $j = 3, \dots, m$. Hence, x_1 dominates all $x \in [\Delta, x_1 + m\epsilon)$. (c) For $\rho = 2, \dots, m-1$ and for all $x \in [x_{\rho-1} + (m-\rho+2)\epsilon, x_\rho + (m-\rho+1)\epsilon)$ we have (i) $P(x, y_j) = P(x_\rho, y_j)$ for $j < \rho-1$, (ii) $P(x, y_{\rho-1}) \leq P(x_\rho, y_{\rho-1})$ (actually, $P(x, y_{\rho-1})$ is constant for $x \in (x_{\rho-1} + (m-\rho+2)\epsilon, x_\rho + (m-\rho+1)\epsilon)$), (iii) $P(x, y_\rho) = P(x_\rho, y_\rho)$, (iv) $P(x, y_{\rho+1}) \leq P(x_\rho, y_{\rho+1})$, since $y_{\rho+1} < x_\rho + (m-\rho+1)\epsilon + \Delta$, (v) $P(x, y_j) = P(x_\rho, y_j)$ for $j \geq \rho+2$. Hence, x_ρ dominates all $x \in [x_{\rho-1} + (m-\rho+2)\epsilon, x_\rho + (m-\rho+1)\epsilon)$, for $\rho = 2, \dots, m-1$. (d) Finally, x_m dominates all $x \in [x_{m-1} + 2\epsilon, X]$. ■

Lemma 3. For Case 1, if Player I is restricted to $\mathcal{X}_{1,m}$, then Player II may restrict his

attention to $\mathcal{Y}_{1,m}$.

Proof: If I is restricted to $\mathcal{X}_{1,m}$, then y_0 dominates all $y \in [0, y_1 - m\epsilon)$, y_1 dominates all $y \in [y_1 - m\epsilon, y_2 - (m-1)\epsilon)$, y_ρ dominates all $y \in (y_\rho - (m-\rho+1)\epsilon, y_{\rho+1} - (m-\rho)\epsilon)$, for $\rho = 2, \dots, m-1$, y_m dominates all $y \in (y_m - \epsilon, 1]$, and finally, $(y_\rho + y_{\rho+1})/2$ dominates $y = y_{\rho+1} - (m-\rho)\epsilon$ for $\rho = 1, \dots, m-1$. ■

In view of Lemma 1, we will be interested in the restricted game $(\mathcal{X}_{1,m}, \mathcal{Y}_{1,m}, P)$, which is a finite game. To describe it, we need some more notation. Let I_m be the unit matrix of dimension $m+1$. Let U_m (respectively, L_m) be the upper ($m+1$) \times ($m+1$) (resp. lower) triangular matrix with 0's along the diagonal and 1's everywhere above (resp. below) the diagonal, and let A_m be the $(m+1) \times (m+1)$ matrix with all its elements 0's except the $(A)_{11} = 1$. Then, Lemmas 1, 2, and 3, lead to

Lemma 4. *For Case 1, the value v of G exists and is the same as the value of the matrix game*

$$M_m^{(1)} := p_{10}(I_m - A_m) + (p_{10} + p_{11})L_m + (p_{10} + p_{00})U_m.$$

Any pair (σ, τ) of optimal strategies in $M_m^{(1)}$ will be optimal in G .

To obtain the value and the optimal strategies for Case 1, let $r := p_{00}/p_{11}$, so that $0 < r < \infty$ from the assumptions $p_{00} > 0$ and $p_{11} > 0$. Let also $R(m) := \sum_{i=0}^m r^i$. We then have

Theorem 1. *If $X \in \left[\frac{2m}{2m+1}, \frac{2m+1}{2m+2}\right)$, for some $m = 1, 2, \dots$ in the game G , then, the value exists and is*

$$v = (p_{10} + p_{11}) \frac{R(m-1)}{\frac{p_{11}}{p_{10} + p_{00}} + R(m-1)}.$$

One optimal strategy of Player I is the probability vector σ on $\mathcal{X}_{1,m}$, giving weights

$$\left(\frac{p_{11}}{p_{10} + p_{00}} + R(m-1)\right)^{-1} \left(\frac{p_{11}}{p_{10} + p_{00}}, 1, \dots, r^{\rho-1}, \dots, r^{m-1}\right)$$

to $x_0, x_1, \dots, x_\rho, \dots, x_m$ and an optimal strategy of Player II is the probability vector τ on $\mathcal{Y}_{1,m}$, giving weights

$$\left(\frac{p_{00}}{p_{10} + p_{11}} + \frac{R(m-1)}{r^{m-1}}\right)^{-1} \left(\frac{p_{00}}{p_{10} + p_{11}}, 1, \dots, r^{-(\rho-1)}, \dots, r^{-(m-1)}\right)$$

to $y_0, y_1, \dots, y_\rho, \dots, y_m$, where (x_ρ, y_ρ) , $\rho = 0, 1, \dots, m$ are defined by

$$\begin{aligned} x_0 &= 0, & x_\rho &= 1 - (2m + 1 - 2\rho)\Delta & \rho &= 1, \dots, m \\ y_0 &= 0, & y_\rho &= 1 - (2m + 2 - 2\rho)\Delta + (m - \rho + 1)\epsilon & \rho &= 1, \dots, m. \end{aligned}$$

Proof: The key in solving the matrix game $M_m^{(1)}$ is to notice that both players possess equalizing strategies. If player I uses $\sigma = (\sigma_0, \sigma_1, \dots, \sigma_m)$, and if $(M_m^{(1)})_{\cdot j}$ denotes the j -th column of $M_m^{(1)}$, then the system

$$\sigma (M_m^{(1)})_{\cdot j} = \sigma (M_m^{(1)})_{\cdot j+1} \quad j = 0, \dots, m - 1$$

has a unique probability distribution as a solution. This satisfies

$$\sigma_\rho = \frac{p_{10} + p_{00}}{p_{11}} r^{\rho-1} \sigma_0 \quad \rho = 1, \dots, m$$

from which the result follows. Similar arguments give the solution for II. ■

Case 2: The open intervals $(\frac{3}{4}, \frac{4}{5}), (\frac{5}{6}, \frac{6}{7}), \dots$

$$X \in \left(\frac{2m+1}{2m+2}, \frac{2m+2}{2m+3} \right), \quad m = 1, 2, \dots$$

Take $\epsilon > 0$ sufficiently small ($\epsilon < 1 - 2(m+1)\Delta$ will do) and define subsets $\mathcal{X}_{2,m} = \{x_\rho, \rho = 0, 1, \dots, m\} \subset \mathcal{X}$ and $\mathcal{Y}_{2,m} = \{y_\rho, \rho = 0, 1, \dots, m\} \subset \mathcal{Y}$ by

$$\begin{aligned} x_m &= X, & x_\rho &= 1 - (2m + 1 - 2\rho)\Delta - \epsilon & \rho &= 0, \dots, m - 1 \\ y_0 &= \Delta - \epsilon, & y_\rho &= 1 - (2m + 2 - 2\rho)\Delta + \epsilon & \rho &= 1, \dots, m. \end{aligned}$$

See Figure 3.

Lemma 5. *For Case 2, if Player II is restricted to $\mathcal{Y}_{2,m}$, then Player I may also restrict his attention to $\mathcal{X}_{2,m}$.*

Proof: If II is restricted to $\mathcal{Y}_{2,m}$, then x_0 dominates all $x \in [0, 2\Delta - \epsilon)$, x_1 dominates all $x \in [2\Delta - \epsilon, x_1 + \epsilon)$, x_ρ dominates all $x \in (x_{\rho-1} + \epsilon, x_\rho + \epsilon)$, for $\rho = 1, \dots, m - 1$, x_m dominates all $x \in (x_{m-1} + \epsilon, X]$, and finally, $(P(x_\rho, y) + P(x_{\rho+1}, y))/2 = P(x_\rho + \epsilon, y)$, for $\rho = 1, \dots, m - 1$, $y \in \mathcal{Y}_{2,m}$. ■

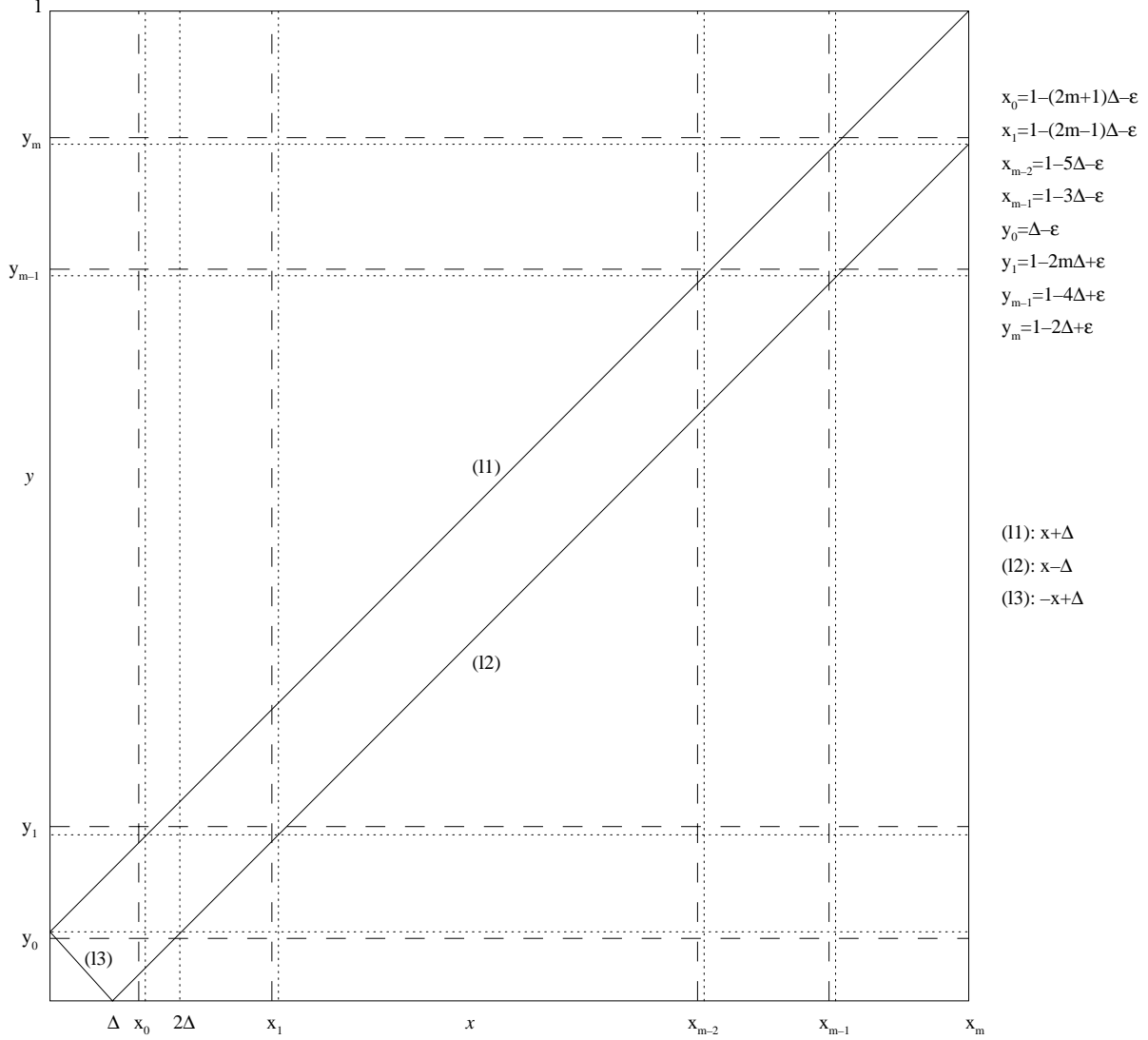


Figure 3. Case 2. The active pure strategies of I and II are represented by the widely spaced dashed lines.

Lemma 6. For Case 2, if Player I is restricted to $\mathcal{X}_{2,m}$, then Player II may also restrict his attention to $\mathcal{Y}_{2,m}$.

Proof: If I is restricted to $\mathcal{X}_{2,m}$, then y_0 dominates all $y \in [0, y_1 - 2\epsilon)$, y_ρ dominates all $y \in (y_\rho - 2\epsilon, y_{\rho+1} - 2\epsilon)$, for $\rho = 1, \dots, m-1$, y_m dominates all $y \in [y_m - 2\epsilon, 1]$, and finally, $(P(x, y_{\rho-1}) + P(x, y_\rho))/2 = P(x, y_\rho - 2\epsilon)$, for $\rho = 1, \dots, m-1$, $x \in X_{2,m}$. ■

Lemmas 1, 5, and 6, lead to

Lemma 7. For Case 2, the value v of G exists and is the same as the value of the matrix

game

$$M_m^{(2)} := p_{10}I_m + (p_{10} + p_{11})L_m + (p_{10} + p_{00})U_m.$$

Any pair (σ, τ) of optimal strategies in $M_m^{(2)}$ will be optimal in G .

Theorem 2. If $X \in \left(\frac{2m+1}{2m+2}, \frac{2m+2}{2m+3}\right)$, for some $m = 1, 2, \dots$, in the game G , then, the value exists and is

$$v = p_{10} + p_{11} - \frac{p_{11}}{R(m)}.$$

An optimal strategy of Player I is the probability vector σ on $\mathcal{X}_{2,m}$, giving weights

$$\frac{1}{R(m)} (1, r, \dots, r^m)$$

to x_0, x_1, \dots, x_m and an optimal strategy of Player II is the probability vector τ on $\mathcal{Y}_{2,m}$, giving weights

$$\frac{1}{R(m)} (r^m, r^{m-1}, \dots, r, 1)$$

to y_0, y_1, \dots, y_m , where (x_ρ, y_ρ) , $\rho = 0, 1, \dots, m$ are defined by

$$\begin{aligned} x_m &= X, & x_\rho &= 1 - (2m + 1 - 2\rho)\Delta - \epsilon & \rho &= 0, \dots, m-1 \\ y_0 &= \Delta - \epsilon, & y_\rho &= 1 - (2m + 2 - 2\rho)\Delta + \epsilon & \rho &= 1, \dots, m. \end{aligned}$$

The proof follows the lines of the proof of Theorem 1 and is omitted.

Case 3: The remaining points $\frac{3}{4}, \frac{5}{6}, \dots$

$$X = \frac{2m+1}{2m+2} \quad m = 1, 2, \dots$$

Subcase 3a: $p_{11} \geq p_{10}$.

The situation for this subcase is similar to that of Case 2, with a slight difference in the strategy spaces of the restricted game. To be precise, the reduced strategy spaces $\mathcal{X}_{3a,m}$ of Player I and $\mathcal{Y}_{3a,m}$ of Player II will be given now by

$$\begin{aligned} x_m &= X, & x_\rho &= 1 - (2m + 1 - 2\rho)\Delta & \rho &= 0, \dots, m-1 \\ y_0 &= \Delta - \epsilon, & y_\rho &= 1 - (2m + 2 - 2\rho)\Delta + \epsilon & \rho &= 1, \dots, m. \end{aligned}$$

Take $\epsilon < \Delta$. Then, assertions identical to those of Lemmas 5 and 6 can be made, the only difference now being that in the proof of the result corresponding to Lemma 6, one has to replace 2ϵ with ϵ . The corresponding matrix game is the same as that of Case 2. For this subcase then, the value of the game $P(x, y)$ and the optimal strategies are exactly as described in Theorem 2.

Subcase 3b: $p_{11} < p_{10}$.

For this subcase the value of $P(x, y)$ *does not* exist for $m \geq 1$, when r is properly defined and non-zero. We show this by showing that the lower value, \underline{v} , and upper value, \bar{v} , of the game are different,

$$\underline{v} = \sup_{\sigma \in \mathcal{X}^*} \inf_{y \in \mathcal{Y}} P(\sigma, y) < \bar{v} = \inf_{\tau \in \mathcal{Y}^*} \sup_{x \in \mathcal{X}} P(x, \tau).$$

Let $M_m^{(3l)}$ denote the matrix game given by

$$M_m^{(3l)} := p_{10}I_m + (1/2)(p_{11} - p_{10})A_m + (p_{10} + p_{11})L_m + (p_{10} + p_{00})U_m.$$

Then we have the following

Lemma 8. For Case 3b, $\underline{v} = \text{val}(M_m^{(3l)})$ $m = 1, 2, \dots$

Proof: To evaluate \underline{v} , we may as well assume that player I announces his strategy σ to player II, before the latter makes his move. But then, successive dominations leave as active strategy spaces

$$\mathcal{X}_{3,m}^{(l)} := \{\Delta, 3\Delta, \dots, (2m+1)\Delta\}$$

for player I, and

$$\mathcal{Y}_{3,m}^{(l)} := \bigcup_{\rho=0}^{m-1} (1 - 2(m - \rho)\Delta, 1 - 2(m - \rho)\Delta + \epsilon) \cup \{0\}$$

for player II. It is easily checked then that the game reduces to the matrix game $M_m^{(3l)}$. ■

We now define the matrix game $(M_m^{(3u)})_{i,j}$ $i=1, \dots, 2m+1$ $j=1, \dots, 2m+2$ in the following way.

$$M_m^{(3u)}_{i,j} = \begin{cases} p_{10} + p_{11} & \text{if } i > j \\ p_{10} + p_{00} & \text{if } i < j - 1 \\ p_{10} & \text{if } i = j - 1 \\ 0 & \text{if } i = j = 1 \\ p_{10} + \frac{1}{2}p_{11} & \text{if } i = j = 2m + 1 \\ p_{10} & \text{if } i = j = 2, \dots, 2m. \end{cases}$$

We may then state

Lemma 9. For Case 3b, $\bar{v} = \text{val}(M_m^{(3u)})$ $m = 1, 2, \dots$

Proof: We may assume that player II announces his mixed strategy τ first. Then, successive simplifications will reduce the strategy spaces to

$$\mathcal{X}_{3,m}^{(u)} := \bigcup_{\rho=0}^{m-1} [((2\rho+1)\Delta - \epsilon, (2\rho+1)\Delta) \cup ((2\rho+1)\Delta, (2\rho+1)\Delta + \epsilon) \\ \cup \{(2m+1)\Delta\}]$$

and

$$\mathcal{Y}_{3,m}^{(u)} := \bigcup_{\rho=0}^m [\{1 - 2(m+1-\rho)\Delta\} \cup \\ (1 - 2(m+1-\rho)\Delta + (m-\rho)\epsilon, 1 - 2(m+1-\rho)\Delta + (m-\rho+1)\epsilon)].$$

Now, the reduced strategy space game can be further simplified to produce $M_m^{(3u)}$, which gives the result. ■

Lemma 10. For Case 3b and for $m \geq 1$, $\bar{v} > \underline{v}$.

Proof: Take $\epsilon > 0$ sufficiently small ($\epsilon < \frac{p_{10}-p_{11}}{2(p_{10}+p_{11}+2p_{00})}$ will do) and suppose that, when playing the game $M_m^{(3u)}$, Player I uses the following, possibly suboptimal, mixed strategy: With probability w_i , $i = 0, \dots, m-1$, he chooses the row pair $(2i+1, 2i+2)$ and with probability w_m he chooses the row $2m+1$, $w_i \geq 0$, $i = 0, \dots, m$, $\sum_{i=0}^m w_i = 1$. Then, given that the pair $(2i+1, 2i+2)$ has been chosen, $i = 0, \dots, m-1$, he chooses the first row of the pair with probability $\frac{1}{2} - I(i=0)\epsilon$ and the second row with probability $\frac{1}{2} + I(i=0)\epsilon$, where I is the indicator function. When playing in this manner, I cannot achieve a payoff greater than $\text{val}(N_m)$, where the matrix game $(N_m)_{i,j}$ $i=0, \dots, m$ $j=1, \dots, 2m+2$ is defined in the following way

$$N_m)_{i,j} = \begin{cases} p_{10} + p_{11} & \text{if } j \leq 2i \\ p_{10} + p_{00} & \text{if } j \geq 2i + 4 \\ p_{10} & \text{if } j = 2i + 2 \\ \frac{p_{10}+p_{11}}{2} + \epsilon(p_{10} + p_{11}) & \text{if } i = 0, j = 1 \\ p_{10} + \frac{1}{2}p_{00} - \epsilon p_{00} & \text{if } i = 0, j = 3 \\ p_{10} + \frac{1}{2}p_{11} & \text{if } i > 0, j = 2i + 1 \\ p_{10} + \frac{1}{2}p_{00} & \text{if } i > 0, j = 2i + 3. \end{cases}$$

Hence,

$$\text{val}(N_m) \leq \text{val}(M_m^{(3u)}) = \bar{v}.$$

But then, for this ϵ , column 2 of N_m is dominated by column 1, column 3 is dominated by the mixture $(1/2)\text{column 1} + (1/2)\text{column 4}$, and, if $m \geq 2$, then, columns $2\rho + 1$, $\rho = 2, \dots, m$ are equal to the mixture $(1/2)\text{column } 2\rho + (1/2)\text{column } 2\rho + 2$, $\rho = 2, \dots, m$. So, if we let

$$M_m^{(3m)}(\epsilon) := M_m^{(3l)} + \epsilon(p_{10} + p_{11})A_m$$

we conclude that

$$\text{val}(N_m) = \text{val}(M_m^{(3m)}(\epsilon))$$

and hence we get

$$\text{val}(M_m^{(3m)}(\epsilon)) \leq \bar{v}.$$

Now, for r well defined and strictly positive, one may evaluate the value function in this equation (in a way similar to that of Theorem 1) and by taking its derivative show that it is strictly increasing in ϵ . Hence,

$$\text{val}(M_m^{(3l)}) < \text{val}(M_m^{(3m)}(\epsilon)).$$

But now, Lemma 8 gives the result. ■

We summarize the discussion of Case 3 with the following.

Theorem 3. *Case 3a: If $X = \frac{2m+1}{2m+2}$ for some $m = 1, 2, \dots$ and $p_{11} \geq p_{10}$ in the game G , then, for (x_ρ, y_ρ) , $\rho = 1, \dots, m$, as found in $\mathcal{X}_{3a,m}$ and $\mathcal{Y}_{3a,m}$, we have the conclusion of Theorem 2.*

Case 3b: If $X = \frac{2m+1}{2m+2}$ for some $m = 1, 2, \dots$ and $p_{11} < p_{10}$ in the game G , then, the value of G does not exist.

Remark. The only cases affected by the choice of the tie rule are Case 2 of Section 2, Case 6 of Section 2 when the saddle point is $(0,0)$ or $(1,1)$, and Case 3 of Section 3 when the value does not exist (Subcase 3b). If the lack of a value is disturbing, it can be avoided by redefining the payoff when there is a tie. Simply call the game a win for

Player I (resp. Player II) when there is a tie. Then the payoff function $P(x, y)$ becomes upper (resp. lower) semi-continuous, and hence the value of $(\mathcal{X}, \mathcal{Y}, P)$ exists by a theorem of K. Fan (1953). (See for example Ferguson (1967) Theorem 2.9.2, or Parthasarathy and Raghavan (1971) Theorem 5.3.5).

If Player I wins all ties, then for X in $[2/3, 3/4)$, $[4/5, 5/6)$, \dots , the value as found in Theorem 1 holds, while for X in $[3/4, 4/5)$, $[5/6, 6/7)$, \dots , the value as found in Theorem 2 holds. If Player II wins all ties, then when X is in $(2/3, 3/4]$, $(4/5, 5/6]$, \dots , the value as found in Theorem 1 holds, while for X in $(3/4, 4/5]$, $(5/6, 6/7]$, \dots , the value as found in Theorem 2 holds. (The optimal strategies may differ.)

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