ON THE INSPECTION GAME

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Abstract. The Inspection Game is a multistage game between a customs inspector and a smuggler, first studied by Melvin Dresher and Michael Maschler in the 1960's. An extension allowing the smuggler to act more than once, treated by Sakaguchi in a special case, is solved. Also, a more natural version of Sakaguchi's problem is solved in the special case where the smuggler may act at each stage.

1. Introduction.

The Inspection Game is a two-person zero-sum multistage game. It was originally proposed by Dresher [5], and treated in greater generality by Maschler [9], in the context of checking possible treaty violations in arms control agreements. This problem may be described as a game between an inspector and a smuggler as follows.

The basic game is played in n stages. Player I, the inspector, chooses k of the stages in which to perform an inspection. Player II, the smuggler, may choose one of the stages to attempt an illegal act. If Player I is inspecting when Player II acts, Player I wins 1 unit and the game ends. If Player II acts when Player I is not inspecting, the payoff is zero. If Player II decides not to act in any of the n stages, Player I wins an amount q between these two values, $0 \leq q \leq 1$. The game is zero-sum so that Player I's winnings are Player II's losses. It is assumed that Player II knows k, and learns of each inspection as it is made. If we denote this game by $\Gamma(n, k)$ with $0 \leq k \leq n$, then we may express this as a finite horizon multistage game with structure

$$\Gamma(n,k) = \begin{array}{l} \\ \text{inspect} \\ \text{don't inspect} \end{array} \begin{array}{c} \begin{array}{cc} \text{act} & \text{wait} \end{array} \\ \begin{pmatrix} 1 & \Gamma(n-1,k-1) \\ 0 & \Gamma(n-1,k) \end{pmatrix} \end{array} \tag{1}$$

for $0 < k < n$, and with boundary conditions for $n \geq 1$,

$$\Gamma(n,0) = (0) \qquad \text{and} \qquad \Gamma(n,n) = (q). \tag{2}$$

The case $q = 1$ may be considered as the case where Player II is required to act at some stage. The problem is to find the value, $V_q(n,k)$, and the optimal strategies of the game $\Gamma(n,k)$.

Simple formulas for the value and optimal strategies were found by Dresher [5] in the case $q = 1/2$ and in the case $q = 1$. Maschler [9] treats the general case as a non-constant-sum game. However in the arms control model he was studying, it is reasonable to assume the inspector may announce his mixed strategy and the other player may not. In his model under these assumptions, both players will choose strategies optimal for the zero-sum game based on the payoff to Player II. Maschler finds these strategies and thus solves the Inspection Game for arbitrary $0 \leq q \leq 1$. His result in our notation may be stated as follows.

Theorem. *(Maschler) For $0 \leq q \leq 1$ and for $0 \leq k \leq n$ and $n \geq 1$, the value of the Inspection Game $\Gamma(n,k)$ is*

$$V_q(n,k) = q\left(1 - \frac{\binom{n-1}{k} q^k}{\sum_{j=0}^{k} \binom{n-k-1+j}{j} q^j}\right) \tag{3}$$

For $0 < k < n$, the optimal mixed strategy for Player I is $(V_q(n,k), 1 - V_q(n,k))$ (unique unless $q = 0$), and the optimal mixed strategy for Player II is $(1-Q, Q)$, where $Q = V_q(n,k)/V_q(n-1,k)$.

This result was obtained independently by Höpfinger [8] but a different form of the formula for the value was obtained. This is

$$V_q(n,k) = q\left(1 - \frac{\binom{n-1}{k}q^k}{\sum_{j=0}^{k}\binom{n}{j}q^j(1-q)^{k-j}}\right). \tag{4}$$

This equivalence of these formulas may be seen as follows. Consider a sequence of independent Bernoulli trials with constant success probability, q. The probability of at most k successes in a negative binomial experiment with $n-k$ failures is equal to the probability of at most k successes in a binomial experiment of n trials. The former probability is $\sum_0^k \binom{n-k-1+j}{j} q^j(1-q)^{n-k}$ as given by the negative binomial distribution, and the latter is $\sum_0^k \binom{n}{j} q^j(1-q)^{n-j}$ as given by the binomial distribution.

Thomas and Nisgav [16] treat this problem when the payoff for being caught is a number $v > 0$, and $q = v$ as well. However, by a rescaling, it may be assumed that $v = q = 1$. One interpretation of such a v is as the probability of being caught when acting during an inspection. Thomas and Nisgav also introduce the possibility of more than one inspection team, each team having its own limited number of days on which it may act, and having possibly different efficiencies, etc. They point out that such problems can be solved as linear programming problems. Baston and Bostock [3] and Garnaev [7] find a closed form of the solution for two and three inspection teams respectively.

In this paper, we treat a generalization of the basic model introduced by Sakaguchi [14] (see also Nakai [10]) in which Player II may act more than once. As an illustration, we may interpret this model through the following scenario. Player II has ℓ truckloads of toxic waste to dispose of. There is no cost for dumping a truckload of toxic waste in the river unless he gets caught by the inspector, in which case he loses +1 each time he is caught. Instead Player II may dispose of any truckload in a legal way at a cost of q per truckload. However, after n days, the inspector will inspect Player I's homebase and force him to dispose legally of any waste found there. In the meantime, he may try to dump one truckload in the river each day. But the inspector only has staff enough to watch him on k of those days.

Von Stengel [17] considers a different generalization in which Player II is allowed to act more than once. In von Stengel's model, the game ends the first time Player II is caught. This provides a better model for the arms control problem. A very interesting feature of von Stengel's solution is that Player I's optimal strategy does not depend on the number ℓ and so is also the solution to the problem when Player I is not informed of Player II's undetected actions. Other interesting papers in this regard are those of Diamond [4] and Ruckle [13]. For a survey of this area, see Avenhaus et al. [1] and [2].

2. Description of the Problem.

Sakaguchi's problem, in which Player II may act more than once, may be extended to arbitrary $0 \le q \le 1$ as follows. If we let ℓ denote the number of times that II may act, and n and k are as before, then, denoting this game by $\Gamma(n,k,\ell)$, the structure may be written

$$\Gamma(n,k,\ell) = \begin{array}{l} \\ \text{inspect} \\ \text{don't inspect} \end{array} \begin{array}{c} \begin{array}{cc} \text{act} \qquad\qquad\qquad & \qquad\qquad \text{wait} \end{array} \\ \begin{pmatrix} 1+\Gamma(n-1,k-1,\ell-1) & \Gamma(n-1,k-1,\ell) \\ \Gamma(n-1,k,\ell-1) & \Gamma(n-1,k,\ell) \end{pmatrix} \end{array} \tag{5}$$

for $1 \le k \le n-1$ and $1 \le \ell \le n-1$. The boundary conditions are

$$\begin{array}{lll} \text{(a)} & \Gamma(n,0,\ell) = (0) & \text{for } 0 \le \ell \le n \\ \text{(b)} & \Gamma(n,k,0) = (0) & \text{for } 0 \le k \le n \\ \text{(c)} & \Gamma(n,n,\ell) = (\ell q) & \text{for } 0 \le \ell \le n \\ \text{(d)} & \Gamma(n,k,n) = (?) & \text{for } 1 \le k \le n-1. \end{array} \tag{6}$$

In the case $q = 1$, it is natural to replace the question mark by k. In Sakaguchi [14], the problem is solved in this case. (A change of location and scale and the interchange of the roles of Players I and II is required to put his problem into the above form.) In Sakaguchi [15], this problem is studied for both $q = 1$ and $q = 1/2$, but for some reason, the question mark in the fourth boundary condition is not specified. Therefore in his Theorem 2 for the case $q = 1/2$, the solution he gives is just one of many possible solutions. It is the one that corresponds to the boundary condition,

$$\Gamma(n,k,n) = (nV_{1/2}(n,k)) \qquad \text{for} \quad 1 \le k \le n-1. \tag{7}$$

where $V_{1/2}(n,k)$ is the value function of the solution for the case $\ell = 1$ and $q = 1/2$ displayed in (3) or (4).

Sakaguchi finds that the value of this game is ℓ times the value when $\ell = 1$, for both $q = 1$ and $q = 1/2$, the latter under the conjecture that all the games have completely mixed strategies. In Lemma 1 below, we show this conjecture is true and in Theorem 2 we show that Sakaguchi's result holds for arbitrary q when the question mark in (6) is replaced by $nV_q(n,k)$.

However, a much more natural way of specifying the values of $\Gamma(n,k,n)$ is to assume that the cost q is always assessed for each of the ℓ actions that is not taken. Since Player II may take at most one action per stage, we have $\Gamma(n,k,n+\ell) = \ell q + \Gamma(n,k,n)$ for all $\ell \ge 0$. Therefore we may specify the $\Gamma(n,k,n)$ indirectly through the recursive equations

$$\Gamma(n,k,n) = \begin{array}{l} \\ \text{inspect} \\ \text{don't inspect} \end{array} \begin{array}{c} \begin{array}{cc} \text{act} \qquad\qquad\qquad & \qquad\qquad \text{wait} \end{array} \\ \begin{pmatrix} 1 + \Gamma(n-1,k-1,n-1) & q + \Gamma(n-1,k-1,n-1) \\ \Gamma(n-1,k,n-1) & q + \Gamma(n-1,k,n-1) \end{pmatrix} \end{array} \tag{8}$$

for $1 \le k \le n-1$, with boundary conditions

$$\Gamma(n,0,n) = (0) \qquad \Gamma(n,n,n) = (nq). \tag{9}$$

The values of these games, found in Theorem 3 below, are the most natural candidates to replace the question mark in (6).

The case $q = 1$ has a close connection to the Simple Point Capture Game (SPCG) of Ruckle [12]. The SPCG is a single-stage game in which the inspector chooses the set of

k days on which to inspect nonsequentially from the n days and the spy similarly chooses the set of ℓ days at the outset on which to act. As in the Inspection Game, the payoff is the number of times the inspector catches the spy acting. The value of the SPCG and the Inspection Game are the same (namely $k\ell/n$, and the unique optimal strategies for the SPCG (namely uniform) are optimal in the Inspection Game as well. The result of Sakaguchi for the Inspection Game with $q = 1$ shows that the players may ignore any information they receive along the way. A similar situation occurs in the game, Hidden Card Goofspiel of Ross [11]. However, some of this has been noted before. The case $q = 1$ of the Inspection Game, Sakaguchi's [14] generalization, Nakai's [9] generalization in his Section 4, and Ruckle's SPCG are all special cases of an earlier general result of Gale [6].

3. The Generalization of Sakaguchi's Problem.

We consider the problem of finding the solution to (5) subject to the boundary conditions (6) with the question mark in (6) replaced by $n\,V_q(n,k)$, where $V_q(n,k)$ is given by (3) or (4).

To solve this problem, we first prove a lemma similar to one conjectured to be true by Sakaguchi. This is used to show that all games $\Gamma(n,k,\ell)$ for $n \geq 2$, $1 \leq k \leq n-1$ and $1 \leq \ell \leq n-1$, are completely mixed, that is that both players have optimal strategies that give positive weight to both actions.

Lemma 1. *For all $n \geq 2$ and $1 \leq k \leq n$, we have*

$$V_q(n,k) - V_q(n,k-1) \leq 1/n.$$

Proof. We rewrite $V_q(n,k)$ using (4) as

$$V_q(n,k) = q\frac{\sum_{j=0}^{k-1}\binom{n-1}{j}q^j(1-q)^{k-1-j}}{\sum_{j=0}^{k}\binom{n}{j}q^j(1-q)^{k-j}} = \frac{\sum_{j=1}^{k}\binom{n-1}{j-1}\left(\frac{1-q}{q}\right)^{k-j}}{\sum_{j=0}^{k}\binom{n}{j}\left(\frac{1-q}{q}\right)^{k-j}}.$$

Now writing $(1-q)/q = x$ and changing variable $t = k - j$ for j, we have

$$V_q(n,k) = \frac{\sum_{t=n-k}^{n-1} \binom{n-1}{t} x^t}{\sum_{t=n-k}^{n} \binom{n}{t} x^t} = \frac{S(\rho, n-1)}{S(\rho, n)}$$

where $\rho = n - k$ and $S(\rho, n) = \sum_{t=\rho}^{n} \binom{n}{t} x^t$. We are to show

$$\frac{S(\rho, n-1)}{S(\rho, n)} - \frac{S(\rho+1, n-1)}{S(\rho+1, n)} \leq \frac{1}{n}$$

for all $x > 0$ and $\rho = 0, 1, \ldots, n-1$. Writing $S(\rho, n-1) = \binom{n-1}{\rho} x^\rho + S(\rho+1, n-1)$ and $S(\rho, n) = \binom{n}{\rho} x^\rho + S(\rho+1, n)$, and multiplying through by $S(\rho, n)S(\rho+1, n)$, we find we are to show

$$n\left[\binom{n-1}{\rho} x^\rho S(\rho+1, n) - \binom{n}{\rho} x^\rho S(\rho+1, n-1)\right] \leq S(\rho, n) S(\rho+1, n) \qquad (10)$$

The left-hand side of (10) is a polynomial in x with terms of the order $x^{2\rho+1}, \ldots, x^{\rho+n}$. The coefficient of $x^{2\rho+s}$ is

$$n\left[\binom{n-1}{\rho}\binom{n}{\rho+s} - \binom{n}{\rho}\binom{n-1}{\rho+s}\right] = s\binom{n}{\rho}\binom{n}{\rho+s} \qquad (11)$$

for $s = 1, \ldots, n-\rho$. The right-hand side of (10) is a polynomial with terms of the order $x^{2\rho+1}, \ldots, x^{2(n-\rho)}$. It is sufficient to show that the coefficients (11) are all less than or equal to the corresponding coefficients on the right side of (10). The coefficient of $x^{2\rho+s}$ on the right side of (10) for $s = 1, \ldots, n-\rho$ is

$$\sum_{j=0}^{s-1} \binom{n}{\rho+j}\binom{n}{\rho+s-j}.$$

The first term cancels one term from the left, so that we are to show

$$(s-1)\binom{n}{\rho}\binom{n}{\rho+s} \leq \sum_{j=1}^{s-1} \binom{n}{\rho+j}\binom{n}{\rho+s-j} \qquad (12)$$

for $s = 2, \ldots, n-\rho$. It is straightforward to show that the ratio, $\binom{n}{\rho}/\binom{n}{\rho+1}$, is increasing in ρ for $0 \leq \rho \leq n-1$. This implies that

$$\binom{n}{\rho}\binom{n}{\rho+s} < \binom{n}{\rho+1}\binom{n}{\rho+s-1}$$

and so on, and therefore, $\binom{n}{\rho}\binom{n}{\rho+s}$ is less than each of the $s-1$ terms of the sum in (12). Thus (12) follows. ∎

Theorem 2. *Let $V_q(n,k,\ell)$ denote the value of the game $\Gamma(n,k,\ell)$. Then,*

$$V_q(n,k,\ell) = \ell V_q(n,k) \qquad \text{for} \quad 0 \le k \le n \quad \text{and} \quad 0 \le \ell \le n. \tag{13}$$

Proof. The boundary conditions (6) give the result for $k=0$, $k=n$, $\ell=0$, and $\ell=n$, for all $n \ge 0$. We must show (13) for all $n \ge 2$, $1 \le k \le n-1$, and $1 \le \ell \le n-1$. The case $n=2$ follows from Theorem 1. As the induction hypothesis, we assume (13) is true with n replaced by $n-1$. Now consider the case n and arbitrary $1 \le k \le n-1$ and $1 \le \ell \le n-1$. From (5) we have

$$V_q(n,k,\ell) = \text{Value}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{14}$$

where

$$\begin{aligned}
\alpha :=& 1 + V_q(n-1,k-1,\ell-1) \\
\beta :=& V_q(n-1,k-1,\ell) \\
\gamma :=& V_q(n-1,k,\ell-1) \\
\delta :=& V_q(n-1,k,\ell).
\end{aligned}$$

The game is completely mixed if $\alpha > \beta$, $\beta < \delta$, $\delta > \gamma$, and $\gamma < \alpha$. The first three of these inequalities follow easily from the induction hypothesis. The last inequality may be written using the induction hypothesis as

$$(\ell-1)V_q(n-1,k) < 1 + (\ell-1)V_q(n-1,k-1). \tag{15}$$

From Lemma 1, (15) holds for all $1 \le \ell \le n-1$ and $1 \le k \le n-1$. Thus the game is completely mixed and (14) reduces to

$$V_q(n,k,\ell) = \frac{\alpha\delta - \beta\gamma}{\alpha - \beta - \gamma + \delta}$$

$$= \frac{(1+(\ell-1)V_q(n-1,k-1))\ell V_q(n-1,k) - \ell V_q(n-1,k-1)(\ell-1)V_q(n-1,k)}{1+(\ell-1)V_q(n-1,k-1) - \ell V_q(n-1,k-1) - (\ell-1)V_q(n-1,k) + \ell V_q(n-1,k)}$$

$$= \frac{\ell V_q(n-1,k)}{1 - V_q(n-1,k-1) + V_q(n-1,k)}.$$

But since $V_q(n,k) = V_q(n,k,1)$, this same equation with $\ell = 1$ shows that the right side is $\ell\, V_q(n,k)$, completing the induction. ∎

4. Solution to (8) and (9).

Theorem 3. *Let $v(n,k)$ denote the value of $\Gamma(n,k,n)$. Then for $0 \le k \le n$,*

$$v(n,k) = k - p^{n-k+1}u(n,k), \tag{16}$$

where $p = 1 - q$ and

$$u(n,k) = \sum_{j=0}^{k-1}(k-j)\binom{n-k-1+j}{j}q^j \tag{17}$$

For $0 < k < n$, the optimal mixed strategy for Player I is (q,p) independent of k and n, and the optimal strategy for Player II is $(Q, 1-Q)$ where $Q = v(n-1,k) - v(n-1,k-1)$.

Proof. Equations (8) and (9) become

$$v(n,k) = \text{Value}\begin{pmatrix} 1 + v(n-1,k-1) & q + v(n-1,k-1) \\ v(n-1,k) & q + v(n-1,k) \end{pmatrix} \qquad \text{for } 1 \le k \le n-1 \tag{18}$$

subject to the boundary conditions

$$v(n,0) = 0 \quad \text{for } n \ge 1, \qquad \text{and} \qquad v(n,n) = nq \quad \text{for } n \ge 0. \tag{19}$$

It is easy to argue directly that $v(n,k-1) \le v(n,k) \le 1 + v(n,k-1)$ for $1 \le k \le n-1$, so the game is completely mixed. Therefore Player I has the optimal mixed strategy (q,p), independent of k and n for $1 \le k \le n-1$, and Player II has the optimal mixed strategy shown. Equation (18) reduces to

$$v(n,k) = q(1 + v(n-1,k-1)) + pv(n-1,k) \qquad \text{for } 1 \le k \le n-1 \tag{20}$$

One may simplify equations (19) and (20) with the change of functions (17) from $v(n,k)$ to $u(n,k)$. Equations (19) and (20) reduce to

$$u(n,k) = qu(n-1,k-1) + u(n-1,k) \qquad \text{for } 1 \le k \le n-1 \tag{21}$$

subject to the boundary conditions

$$u(n,0) = 0 \quad \text{for } n \ge 1, \quad \text{and} \quad u(n,n) = n \quad \text{for } n \ge 0. \tag{22}$$

The solution is given by (16) for $0 \le k \le n$. This is easily seen by checking that it satisfies (21) and (22). ■

It is interesting that Player I's strategy is independent of k and n provided $k < n$. For example, suppose that $n = \ell = 20$, $k = 1$ and $q = 1/2$. Usually if an action can be carried out only once, it is important not to carry it out too soon, since it gives the opponent free rein afterward. This is the basis of the maxim "The threat is more powerful than the execution". But in this example, it is optimal for player I to carry out his one and only action half the time at the very first stage out of twenty stages.

For $0 < q < 1$, it is an open problem to find a nonrecursive solution to $\Gamma(n,k,\ell)$ subject to (5) and (6) with the question mark in (6) replaced by (16). The case $q = 1$ is Sakaguchi [14] or Theorem 2.

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