Some Chip Transfer Games

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Abstract: Proposed and investigated are four impartial combinatorial games: Empty & Transfer, Empty-All-But-One, Empty & Redistribute, and Entropy Reduction. These games involve discarding chips from some boxes and transferring chips from one box to another. Empty & Transfer and Entropy Reduction are solved only for a small number of boxes. Empty-All-But-One, Empty and Redistribute and Entropy Reduction are solved in both normal and misère forms. The Sprague-Grundy function for Empty & Transfer is found in the two box case.

Keywords: Impartial combinatorial game, Sprague-Grundy function, misère play.

1. Introduction.

Four related impartial combinatorial games are investigated. The first game is called *Empty & Transfer*. Given k boxes each containing at least one chip, a move consists of emptying a box and transferring to it at least one chip, but not all chips, from one of the other boxes. Players move alternately and play continues until no more moves are possible. Under the normal ending rule, the last player to move wins; under the misère rule, the last player to move loses. The normal form of this game is solved for $k \leq 4$. For $k = 2$, the Sprague-Grundy function is found. The second game is called *Empty-All-But-One*. A move consists of emptying all but one of the boxes and transferring chips from the remaining box so that there is at least one chip in each box. The third game is called *Empty & Redistribute*. A move empties only one box but the chips in the remaining boxes may be redistributed among all boxes in an arbitrary manner. These two games are solved for arbitrary k in both normal and misère forms. The last game is called *Entropy Reduction*. Two boxes are chosen and made more nearly equal in size by transferring chips from one box to the other. This game is solved for $k = 3$ in both the normal and the misère versions.

Solving these games involves the standard procedure of finding the P-positions and N-positions. A position is a P-position if the Previous player (the one who just moved) can win with optimal play. It is an N-position if the Next player to move can win with optimal play. We use $\mathcal{P}$ to denote the set of P-positions, $\mathcal{N} = \mathcal{P}^c$ to denote the set of N-positions and $\mathcal{T}$ to denote the set of terminal positions. After conjecturing a set $\mathcal{P}$ to be the set of P-positions, one checks whether the conjecture is true by checking the following three conditions.

1. For normal play, $\mathcal{T} \subset \mathcal{P}$. For misère play, $\mathcal{T} \subset \mathcal{N}$.
2. For any position $p \in \mathcal{P}$ and for any position q that can be reached from p in one move, we have $q \in \mathcal{N}$.
3. For any position $p \in \mathcal{N}$ that is not terminal, there exists a move to a position $q \in \mathcal{P}$.

There are various related problems in the literature, many of which are described in the basic book on combinatorial games by Berlekamp, Conway and Guy (1982). A related chip transfer game, called Fulves's Merger, is described in Guy (1995) (see unsolved problem #38). In this game, any number of chips (possibly all) may be transferred from one box to another box whose size is at least as great as the original box. The player who first makes all boxes contain an even number of chips is the winner. (The total number of chips is assumed even.)

In general, games of this sort, in which the sizes of two or more boxes may change simultaneously in one move, may not be written as a disjunctive sum of games. Examples are Moore's Nim_k in which as many chips as desired may be removed from up to k boxes (see, for example, Jenkens and Mayberry (1980)), Matrix Nim_k of Holladay (1958), and Wythoff's Nim (see, for example, Fraenkel and Lorberbom (1995)).

Another example may be found in the games Trim and Rim of Jim Flanigan (1980). Since this interesting paper is unpublished, I mention the rules of these games. However, I leave it to the reader to discover their secret which involves the notion of trim-sum (addition without carry in base 3) and rim_k-sum. In Trim, a player may remove any positive number, x, of chips from any box, and also, if desired, he may remove exactly y chips from another box, provided y is a power of three and $y > x/2$. Rim_k for $k \geq 2$ is a generalization of Nim and Trim, with $\mathrm{Rim}_2 = \mathrm{Nim}$ and $\mathrm{Rim}_3 = \mathrm{Trim}$. In Rim_k, a player may remove any positive number, x, of chips from any box, and also he has the option, that may be exercised up to $k-2$ times in any one move, of removing precisely y chips from any box (including the box from which the x chips were taken, and possibly using the same box more than once), provided y is a power of k and $y > x/2$. Flanigan uses as an illustration the game of Rim_5 played with just 5 boxes of sizes 71, 135, 138, 176, and 252. One winning move for the first player is to remove 62 chips from the box of size 71, and 50 chips from the box of size 176. This is effected by setting $x = 37$ taken from the first box and then exercising the option three times with $y = 25$, once from the first box and twice from the fourth box.

There are many unsolved problems in connection with the Chip Transfer games. Empty & Transfer has been solved only for $k \leq 4$. Can one find simple solutions for any $k \geq 5$? Similarly, no simple description of the solution of the misére version of Empty & Transfer has been found for any $k \geq 3$. Entropy Reduction has only been solved for $k \leq 3$. Moreover, the Sprague-Grundy function has not been found for any of these games for any $k \geq 3$. It might also be interesting to find suspense and remoteness numbers for these games.

There are many possible generalizations of these games that are worth investigating. However, the most promising of these seems to be one suggested by a referee that bridges Empty-All-But-One and Empty & Redistribute. It is played with k nonempty boxes, and a move consists in emptying j boxes (where $j < k$ is a fixed number) and redistributing the rest so that no box is empty. A similar game, in which the mover may select the number j of boxes to empty, is easy and is described in Section 4.

2. Empty & Transfer.

Rules: There are $k \geq 2$ boxes each containing a positive integer number of chips. A move of the game consists of emptying one of the boxes, selecting another box and transferring some but not all of the chips from it into the empty box. Last player to move wins.

There is a unique terminal position, namely $\mathcal{T} = \{(1, 1, \ldots, 1)\}$. It seems difficult to solve this game for general k, but we have one general result.

Proposition 1. *Let $\mathcal{O}$ denote the set of positions $(n_1, \ldots, n_k)$ such that all n_i are odd. Then $\mathcal{O} \subset \mathcal{P}$.*

Proof. Clearly $\mathcal{T} \subset \mathcal{O}$. Any move from a position in which all of the components are odd must be to a position of the form with exactly one even component. But we may immediately put this position back into $\mathcal{O}$ by emptying any odd box and putting one chip from the even box into it. ∎

Proposition 2. *Any position with exactly one or two even components is an N-position.*

Proof. Consider such a position. If it has two even components, empty one of them; if it has one even component then empty an odd component. In either case, place one chip from the remaining even component in the empty box. This leaves a position with all odd components, which is a P-position by Proposition 1. ∎

As a corollary, we note that we have solved the case $k = 2$. In this case we have $\mathcal{P} = \mathcal{O}$, since positions with all components odd are P by Proposition 1, and the rest of the positions are N by Proposition 2.

For $k = 2$ we can find the Sprague-Grundy function. This can be used to solve a sum of such games.

Proposition 3. *For $k = 2$, the Sprague-Grundy function, $g(x, y)$, may be found inductively as*

$$g(x, y) = 0 \quad \textit{if both } x \textit{ and } y \textit{ are odd}$$
$$g(2x, 2y) = g(2x, 2y - 1) = g(2x - 1, 2y) = 1 + g(x, y).$$

For example, the SG-value of the position $(31, 54)$ is $g(31, 54) = 1 + g(16, 27) = 2 + g(8, 14) = 3 + g(4, 7) = 4 + g(2, 4) = 5 + g(1, 2) = 6 + g(1, 1) = 6$.

Another method of computing the Sprague-Grundy value is useful for thinking about the proof of Proposition 3. Expand x and y in binary: $x = (\cdots x_2 x_1 x_0)_2$ and $y = (\cdots y_2 y_1 y_0)_2$. Find the smallest j such that not both $x_j = 0$ and $y_j = 0$. If both $x_j = 1$ and $y_j = 1$, then $g(x, y) = j$; otherwise, add 2^j to x if $x_j = 1$ and add it to y if $y_j = 1$. This does not change the g value of the pair. Now apply the same process again to obtain a larger j, and continue this process until $x_j = y_j = 1$. Then $g(x, y) = j$. We illustrate this method on the same example.

$$\begin{cases} 31 = (011111)_2 \\ 54 = (110110)_2 \end{cases} \to \begin{cases} 32 = (100000)_2 \\ 54 = (110110)_2 \end{cases} \to \begin{cases} 32 = (100000)_2 \\ 56 = (111000)_2 \end{cases} \to \begin{cases} 32 = (0100000)_2 \\ 64 = (1000000)_2 \end{cases}$$
$$\to \begin{cases} 64 = (1000000)_2 \\ 64 = (1000000)_2 \end{cases}.$$

Thus $g(x, y) = 6$, since the 1's occur in the sixth column from the right starting the counting at 0. From this method for computing g, we immediately see that if $g(x, y) = n$, then $g(u, v) = n$ for all (u, v) such that $u \equiv x \pmod{2^{n+1}}$ and $v \equiv y \pmod{2^{n+1}}$.

To prove proposition 3, we use three lemmas.

Lemma 1. *Suppose $g(u,v)=n$. Then $a=u \pmod{2^{n+1}}$ and $b=v \pmod{2^{n+1}}$ satisfy*

$$1 \le a \le 2^n \quad 1 \le b \le 2^n \quad \textit{and} \quad 2^n+1 \le a+b \le 2^{n+1}.$$

Proof. By induction: The result is true for $n=0$, since $g(u,v)=0$ means that both u and v are odd, so that $a=b=1$.

Now suppose the result is true for $n-1$. If $g(u,v)=n$, then $u=2x$ or $u=2x-1$ for some (x,y) such that $g(x,y)=n-1$. By the induction hypothesis, $a=x \pmod{2^n}$, $b=y \pmod{2^n}$ satisfy $1 \le a \le 2^{n-1}$ and $1 \le b \le 2^{n-1}$. Hence if $c=u \pmod{2^{n+1}}$, then $c=2a$ or $c=2a-1$, so that $1 \le c \le 2^n$. Similarly if $d=v \pmod{2^{n+1}}$, then $d=2b$ or $d=2b-1$. In addition, $a+b=2(c+d)$ or $u+v=2(c+d)-1$, so that $2^n+1 \le a+b \le 2^{n+1}$. ■

Lemma 2. *For every x such that $2^n+1 \le x \le 2^{n+1}$, there exists (u,v) such that $g(u,v)=n$ and $u+v \equiv x \pmod{2^{n+1}}$.*

Proof. In fact, we have $g(u,v)=n$ for $u=2^n$ and any v such that $1 \le v \le 2^n$ as is easily seen by the binary computation method. ■

Lemma 3. *If j is such that both $1 \le x \pmod{2^{j+1}} \le 2^j$ and $1 \le y \pmod{2^{j+1}} \le 2^j$, then $g(x,y) \le j$.*

Proof. Consider the binary expansion, $x=(\cdots x_2x_1x_0)_2$ and $y=(\cdots y_2y_1y_0)_2$. If $x_j=1$ (similarly $y_j=1$), then, since $x_i=0$ for $i<j$, the binary method of computing g shows that $g(x,y)=j$. Now suppose that $x_j=y_j=0$. Now the binary method may stop before it changes x_j or y_j to 1, in which case $g(x,y)<j$. Otherwise, if x_j (similarly y_j) changes to 1, then all $x_i=0$ for $i<j$, so that again the binary method shows that $g(x,y)=j$. In all cases, $g(x,y) \le j$. ■

Proof of Proposition 3. Let $n \ge 0$ and assume that $g(u,v)=n$. We must show (1) for all $j<n$, there is a move from (u,v) into some point (x,y) such that $g(x,y)=j$, and (2) there is no move from (u,v) into any (x,y) such that $g(x,y)=n$.

(1) Suppose that $j<n$. Let $a=u \pmod{2^{j+1}}$ and $b=v \pmod{2^{j+1}}$. First note that we cannot have both $1 \le a \le 2^j$ and $1 \le b \le 2^j$, because Lemma 3 would imply that $g(u,v) \le j$, contradicting $g(u,v)=n$. So we have at least one of a and b is either 0 or greater than 2^j (but less than 2^{j+1}). Suppose it is a. Then by Lemma 2, the move (u,v) into $(x,y)=(u-2^j,2^j)$ has $g(x,y)=j$.

(2) Now let $a=u \pmod{2^{n+1}}$ and $b=v \pmod{2^{n+1}}$. By Lemma 1, we have $1 \le a \le 2^n$ and $1 \le b \le 2^n$. But any move of (u,v) into (x,y) has either $x+y \equiv u \pmod{2^{n+1}}$ or $x+y \equiv v \pmod{2^{n+1}}$, and in either case we would have $1 \le x+y \pmod{2^{n+1}} \le 2^n$, and, by Lemma 1 again, we would not have $g(x,y)=n$. ■.

We can also solve completely the cases $k=3$ and $k=4$.

Proposition 4. *In the three box case, $\mathcal{P}=\cup_0^\infty \mathcal{P}_j$, where*

$$\mathcal{P}_j=\{(n_1,n_2,n_3) : \textit{for all } i,\ n_i=2^j m_i \textit{ for some odd number } m_i\}.$$

Proof. 1. The terminal position is in $\mathcal{P}$.

2. If $\mathbf{n} = (n_1, n_2, n_3) \in \mathcal{P}_j$, then emptying one box and transfering chips to it from another will leave 2^j times an odd number in the remaining box, but there is no way to write $2^j m$ for m odd as a sum of two numbers of this form. Thus, no move from $\mathcal{P}_j$ can be in $\mathcal{P}$.

3. If $\mathbf{n} \notin \mathcal{P}$, then for each i find j_i such that $n_i = 2^{j_i} m_i$ for m_i odd. Suppose without loss of generality that $j_1 \le j_2 \le j_3$. Then $j_1 < j_3$ since $\mathbf{n} \notin \mathcal{P}$. Therefore we may empty box 2, and put 2^{j_1} chips from box 3 into box 2. The resulting position is in $\mathcal{P}_{j_1} \subset \mathcal{P}$. ■

We now treat the 4 box case. All P-positions may be found using the following two propositions. These propositions are not valid for a larger number of boxes.

Proposition 5. *$(a, b, c, d) \in \mathcal{P}$ if, and only if, $(2a, 2b, 2c, 2d) \in \mathcal{P}$.*

Proof. If the result is not true, there exists a smallest counterexample in the sense that the sum $a + b + c + d$ is smallest. Take such a counterexample and suppose first that $(a, b, c, d) \in \mathcal{P}$ and $(2a, 2b, 2c, 2d) \in \mathcal{N}$. Then $(2a, 2b, 2c, 2d)$ can be moved into $\mathcal{P}$. Suppose without loss of generality, that it is moved to $(x, y, 2c, 2d) \in \mathcal{P}$ where $x+y = 2a$. By Proposition 2, not both x and y can be odd. Hence both x and y are even and we can write $x = 2u$ and $y = 2v$. But $(u, v, c, d) \in \mathcal{N}$ since it can be reached in one move from $(a, b, c, d) \in \mathcal{P}$. Then $(u, v, c, d) \in \mathcal{N}$ and $(2u, 2v, 2c, 2d) \in \mathcal{P}$ gives a smaller counterexample.

Now suppose the smallest counterexample has $(a, b, c, d) \in \mathcal{N}$ and $(2a, 2b, 2c, 2d) \in \mathcal{P}$. There must be a move putting (a, b, c, d) into $\mathcal{P}$, say to $(a', b', c', d') \in \mathcal{P}$. But the same move (using twice as many chips everywhere) puts $(2a, 2b, 2c, 2d) \in \mathcal{P}$ to $(2a', 2b', 2c', 2d') \in \mathcal{N}$. This also gives a smaller counterexample. ■

Proposition 6. *$(a, b, c, d) \in \mathcal{P}$ if, and only if, $(2a, 2b, 2c, 2d - 1) \in \mathcal{P}$.*

Proof. If the result is not true, there exists a counterexample with smallest sum $a + b + c + d$. First suppose such a counterexample has $(a, b, c, d) \in \mathcal{P}$ and $(2a, 2b, 2c, 2d-1) \in \mathcal{N}$. There exists a move from $(2a, 2b, 2c, 2d-1)$ into $\mathcal{P}$. This move cannot empty the box with an odd number of chips, because the same move could get rid of $2d$ from $(2a, 2b, 2c, 2d)$, which from Proposition 5 is in $\mathcal{P}$. Therefore there must be an odd number of chips left after the move, so we may suppose that the move is to a point $(2r, 2s, 2t, 2u-1) \in \mathcal{P}$. But then $(a, b, c, d) \in \mathcal{P}$ into $(r, s, t, u) \in \mathcal{N}$ is a legal move. This gives a counterexample with a smaller sum.

Now suppose the smallest counterexample has $(a, b, c, d) \in \mathcal{N}$ and $(2a, 2b, 2c, 2d-1) \in \mathcal{P}$. Then there exists a move from (a, b, c, d) to $\mathcal{P}$. If such a move empties the box of d chips, the corresponding move from $(2a, 2b, 2c, 2d - 1)$ with twice as many chips, gives a counterexample to Proposition 5. Similarly, if one of a, b or c, say a, was removed and then b or c was redistributed, the corresponding move would be available from $(2a, 2b, 2c, 2d-1)$ with twice as many chips, giving a smaller counterexample. Finally, if one of a, b or c, say a, was removed and then d was redistributed, The corresponding move from $(2a, 2b, 2c, 2d-1)$, with one less chip in one of the cells to which the d was moved gives a smaller counterexample. Thus in every case there is a smaller counterexample. ■

From these propositions, it is easy to determine the outcome of an arbitrary position. Given a position (a, b, c, d), we may repeatedly divide by 2 until there is at least one odd number without changing the outcome (Proposition 5). If there are then 4 odd numbers, the position is a P-position (Proposition 1). If there are 3 or 2 odd numbers, the position is an N-position (Proposition 2). If there is exactly one odd number, then adding 1 to it does not change the outcome (Propositions 5 and 6), but then all numbers are even and the method may be repeated until the outcome is resolved.

An optimal move from an N-position may also be found easily. Let $g(a,b,c,d)$ denote the number of divisions by 2 in the above algorithm, similar to the SG-function for the case $k=2$. If the algorithm ends with 3 odd numbers, then empty any of the 3 boxes that led to an odd number and transfer $2^{g(a,b,c,d)}$ chips to it from the box that led to the even number. If the algorithm ends with 2 odd numbers, then empty either of the boxes that led to an even number and transfer $2^{g(a,b,c,d)}$ chips to it from the other box that led to an even number. The maximum number of arithmetic operations needed to carry out these computations is linear in the logarithm of the maximum number of chips in a box.

It is surprising that the same type of numerical operations involved in finding the Sprague-Grundy values of the game for $k=2$ are used for $k=4$ to find the outcome.

3. Empty-All-But-One.

Rules: There are k boxes each containing a positive integer number of chips. A move of the game consists of emptying $k-1$ of the boxes and redistributing the contents of the remaining box among all the boxes in such a way that there is at least one chip in each box. Note that for $k=2$, Empty-All-But-One is the same as Empty & Transfer, and so the normal form of it has been solved in Section 2.

The terminal positions are those in which all coordinates are positive and less than k:

$$\mathcal{T} = \{\mathbf{n} = (n_1,\ldots,n_k) : 1 \le n_i \le k-1 \quad \text{for } i = 1,\ldots,k\}.$$

Proposition 7. *The set of P-positions is*

$$\mathcal{P} = \{\mathbf{n} = (n_1,\ldots,n_k) : 1 \le n_i \pmod{k(k-1)} < k \quad \textit{for } i = 1,\ldots,k\}.$$

Proof. 1. Clearly all terminal positions are in $\mathcal{P}$.

2. If $\mathbf{n} \in \mathcal{P}$ is moved to $\mathbf{m} = (m_1\ldots,m_k)$, where $m_1+\cdots+m_k = n_i$ for some i, then it cannot be true that we have $1 \le m_j \pmod{k(k-1)} < k$ for all j, because then $n_i = m_1+\cdots+m_k$ would be between k and $k(k-1) \pmod{k(k-1)}$ contradicting $\mathbf{n} \in \mathcal{P}$. Thus $\mathbf{m} \notin \mathcal{P}$.

3. If $\mathbf{n} \notin \mathcal{P}$, then for some i we have $n_i \pmod{k(k-1)}$ is between k and $k(k-1)$. Any such n_i may be written as a sum $m_1+\cdots+m_k$ where all $m_j \pmod{k(k-1)}$ are between 1 and k. Therefore there exists a move of $\mathbf{n}$ into $\mathbf{m} \in \mathcal{P}$. ■

We may also solve the misère version of the game for general k. In the misère version of the game, the terminal positions are N-positions.

Proposition 7(misère). *Let $q = (k-1)(2k-1)$. The set $\mathcal{P}$ of P-positions for the misère version of Empty-All-But-One consists of those positions $\mathbf{n} = (n_1,\ldots,n_k)$ such that for all i, either (1) $1 \le n_i \le k-1$, or (2) $k \le n_i \pmod{q} \le 2k-2$, but not all $n_i < k$.*

$$\mathcal{P} = \{(n_1,\ldots,n_k)) : \textit{for all } i, \textit{ either } 1 \le n_i \le k-1 \textit{ or } k \le n_i \pmod{q} \le 2k-2\} - \mathcal{T}.$$

Proof. 1. None of the terminal positions is in $\mathcal{P}$, since we have excluded the case of all $n_i < k$.

2. Suppose $\mathbf{n} \in \mathcal{P}$ is moved to $\mathbf{m} = (m_1, \ldots, m_k)$, where $m_1 + \cdots + m_k = n_i$ for some i. Then n_i must satisfy $k \leq n_i \pmod{q} \leq 2k-2$. But the sum modulo q of the coordinates of any P-position is at least $(k-1)+k = 2k-1$ and at most $k(2k-2) = 2k(k-1) = k-1$ modulo q. That is, no P-position has the sum of coordinates between k and $2k-2$ modulo q. Thus $\mathbf{m}$ is not in $\mathcal{P}$.

3. If $\mathbf{n}$ is not in $\mathcal{P}$ and is not terminal, then it has at least one coordinate $n_i > k-1$ such that $n_i \pmod{q}$ is not between k and $2k-2$. But we can achieve any number between $(k-1)+k$ and $k(2k-2)$ as a sum of k numbers between 1 and $2k-2$ with at least one of them at least k. Hence, we may empty all but box i, and distribute these n_i chips in all boxes so that the resulting position lies in $\mathcal{P}$. ■

The proofs of these propositions are constructive and indicate how to find a P-position from a given N-position in a number of operations that is linear in k.

4. Empty & Redistribute.

Rules: There are k boxes, each containing a positive number of chips. A move in the game consists of two parts. First, one of the boxes is chosen and all chips are removed from it and discarded. Second, the chips remaining in the other $k-1$ boxes are redistributed among the k boxes in such a way that there is at least one chip in each box. Players alternate moves and the last player to move wins.

For $k = 2$, this game is the same as Empty & Transfer and Empty-All-But-One. We solve this game for arbitrary $k \geq 2$. A position in this game may be represented by a k-tuple, $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ where x_i, for each $i = 1, 2, \ldots, k$, is a positive integer representing the number of chips in the ith box. We denote the set of all positions by $\mathcal{X}_k$. We denote the sum of the coordinates of $\mathbf{x}$ by $|\mathbf{x}| = x_1 + x_2 + \cdots + x_k$ and refer to $|\mathbf{x}|$ as the norm of $\mathbf{x}$. Since the ordering of the boxes makes no difference, we may assume that $x_1 \leq x_2 \leq \ldots \leq x_k$ in this representation. We also use the multiplicative notation to represent positions, with a power indicating the number of times a particular coordinate value occurs. Thus, $1^3 2\, 3^2 5$ represents the position $(1,1,1,2,3,3,5) \in \mathcal{X}_7$. There is a unique terminal position, namely $1^k = (1,1,\ldots,1)$.

This game has a feature that enables one to play it optimally without knowing all the P-positions. (This is fortunate because it is difficult to describe all P-positions explicitly. This is illustrated in the case $k = 7$ after the proof of the main result). In fact, one only needs to know the set, called $S(k)$ below, of positive integers that are achievable as norms of P-positions. Then a position, $\mathbf{x}$, is an N-position if there is at least one coordinate, x_i, such that $|\mathbf{x}| - x_i \in S(k)$, and otherwise $\mathbf{x}$ is a P-position. Below we describe $S(k)$ and give for each $n \in S(k)$ a P-position of the simple form $\mathbf{x}(n) = v^{k-1}z$ for some integers v and z. Then an optimal move at an N-position, $\mathbf{x}$, is to find a coordinate x_i such that $n = |\mathbf{x}| - x_i \in S(k)$ (if none exists, then $\mathbf{x}$ is an P-position), and to discard x_i and to redistribute the remaining chips into $\mathbf{x}(n)$.

The idea behind this solution can be illustrated in the much simpler game called *Selective-Empty & Redistribute.* In this game, a move consists of emptying any number j, $1 \leq j \leq k-1$, of boxes and redistributing the contents of the remaining boxes arbitrarily among the k boxes so that no box is empty. Here, the set, $S(k)$, of norms of P-positions has the simple form $S(k)=\{n\geq k: n=0 \pmod{k}\}$. Define for each $n\in S(k)$, $\mathbf{x}(n)=1^{k-1}z$ where $z=n-(k-1)$. To show that $S(k)$ is the set of integers achievable as norms of P-positions, and that each $\mathbf{x}(n)\in\mathcal{P}$, it is sufficient to show (1) $\mathcal{T}\subset\{\mathbf{x}(n): n\in S(k)\}$, (2) no $\mathbf{x}(n)$ with norm in $S(k)$ can be moved to a position with norm in $S(k)$, and (3) every position with norm not in $S(k)$ can be moved into some $\mathbf{x}(n)$. (1) follows since $\mathcal{T}=\{\mathbf{x}(k)\}$. (2) follows since the only moves from $\mathbf{x}(n)$ are to empty some number j of the boxes containing 1 chip. Since $1\leq j\leq k-1$, this changes the modulus of the norm to a non-zero value. Finally, to show (3), we must show that for any integers $x_1,x_2,\ldots,x_k$ there is a nonempty subset of them whose sum is $0 \pmod{k}$. This is a well-known result; see, for example Roberts (1984), Exercise 8.1.26. (In fact, from this exercise, we can see that the same result and solution holds if the boxes are ordered and it is required that the boxes to be emptied are consecutive.) Note that there are P-positions with norm in $S(k)$ other than the $\mathbf{x}(n)$. For example, when k is odd, 2^k is a P-position.

In the game Empty & Redistribute, the structure of the solution depends on an integer function of k defined as follows.

$$f(k)=\min\{j\geq 1: k-1 \text{ is not divisible by } j+1\}.$$

Note that $f(k)=1$ if and only if k is even, and for example $f(7)=3$ and $f(61)=6$. There is no k such that $f(k)=5$.

We describe the set $S(k)$ by describing the complement set $R(k)$. Define the sets $R_j(k)$ for $j=1,\ldots,f(k)$ as follows.

$$R_j(k)=\begin{cases}\{n: jk\leq n<(j+1)k \quad\text{and}\quad n=j(k-1) \pmod{(j+1)}\} & \text{if } j<f(k)\\ \{n: n\geq kf(k) \quad\text{and}\quad n=f(k)(k-1) \pmod{(f(k)+1)}\} & \text{if } j=f(k)\end{cases}$$

For example, $R_1(6)=\{7,9,11,\ldots\}$ and $R_2(7)=\{15,18\}$. Then define

$$R(k)=\cup_{j=1}^{f(k)}R_j(k) \qquad\text{and}\qquad S(k)=\{k,k+1,k+2,\ldots\}-R(k).$$

For example, $R(7)=\{8,10,12\}\cup\{15,18\}\cup\{22,26,30,\ldots\}$.

A useful property of $f(k)$ is expressed in the following lemma.

Lemma 4. *If $j\leq f(k)$, then $(k-1)j\notin S(k)$.*

Proof. If $j\leq f(k)$, then $(k-1)$ is divisible by j. The largest element of $R_{j-1}(k)$ is $(j-1)(k-1)+j(k-1)/j=(k-1)j$. Thus, $(k-1)j$ is in $R_{j-1}(k)$, and hence is not in $S(k)$. ■

For each $n\in S(k)$, we associate a position, $\mathbf{x}(n)$ as follows. If $n<f(k)k$, find j such that $jk\leq n<(j+1)k$; otherwise (if $n\geq f(k)k$), let $j=f(k)$. Then $1\leq j\leq f(k)$. Now let $v=n-j(k-1) \pmod{(j+1)}$, $0\leq v\leq j$. Note that since $n\in S(k)$, then $v\neq 0$ because $n=j(K-1) \pmod{(j+1)}\in R_j(k)$ and so has been excluded from $S(k)$. Thus, $1\leq v\leq j$. Now define $\mathbf{x}(n)$ as

$$\mathbf{x}(n)=v^{k-1}z \qquad\text{where}\quad z=n-(k-1)v.$$

As an example, suppose $k = 7$; then $f(k) = 3$ and $S(k) = \{7, 9, 11, 13, 14, 16, 17, 19, 20, 21, 23, 24, 25, 27, 28, 29, \ldots\}$. For $n = 7$, we have $j = 1$, $v = 7 - 7 - 1 \pmod 2 = 1$ and $z = 1$, so $\mathbf{x}(7) = 1^6 1 = 1^7$, the terminal position. For $n = 16$, we have $j = 2$, $v = 1$ and $z = 10$, so $\mathbf{x}(16) = 1^6 10$. Similarly, $\mathbf{x}(28) = 2^6 16$.

General Strategy. Given an N-position, $\mathbf{x}$, find a component x_i such that $n = |\mathbf{x}| - x_i \in S(k)$. (If no such x_i exists, $\mathbf{x}$ is a P-position.) Remove x_i chips and move to $\mathbf{x}(n)$.

The number of arithmetic operations required is linear in k: k additions to find the norm of $\mathbf{x}$, plus many fewer than k to find $f(k)$, plus k divisions to find a move from an N-position. The main result is that this general strategy always works.

Proposition 8. *For all $n \in S(k)$, $\mathbf{x}(n)$ is a P-position. Every position $\mathbf{x}$ of norm $n \in R(k)$ is an N-position. The proof is constructive and indicates how a P-position may be found from a given N-position in a number of operations that is linear in k.*

Proof. It is sufficient to show that any move from a position reached by the strategy can be reversed by a move back into one of these positions. Then eventually the strategy will move into the terminal position, $\mathbf{x}(k)$. We first show that every move from one of the positions $\mathbf{x}(n)$ must be to a position with norm in $R(k)$. Then we show that every position with norm in $R(k)$ can be moved into a position with norm in $S(k)$ and hence into one of the $\mathbf{x}(n)$.

Suppose $n \in S(k)$. From any position of the form $\mathbf{x}(n) = v^{k-1}z$ there are only two types of moves, one removing v chips and the other removing z chips. Removing z chips leads to a position with norm $(k-1)v$. Since all positions $\mathbf{x}(n)$ with $n \in S(k)$ are of this form with $1 \le v \le f(k)$, Lemma 4 shows that the resulting position has norm in $R(k)$. Suppose therefore that v chips are removed from $\mathbf{x}(n) = v^{k-1}z$ where $z > v$. If $n > f(k)k$, then n is of the form $n = kf(k) + 1 + m(f(k)+1) + v$, with $m \ge 0$ and $1 \le v \le f(k)$. Removing v chips leaves $f(k)(k-1) + (f(k)+1)(m+1)$ chips which is not in $S(k)$. Similarly for $n < f(k)k$: Find j such that $jk < n < (j+1)k$ and write n in the form $n = jk + 1 + m(j+1) + v$ with $0 \le m \le (k-1)/(j+1)$ and $1 \le v \le j < f(k)$. Removing v chips leaves $j(k-1) + (j+1)(m+1)$ chips which is not in $S(k)$.

Now suppose $n = |\mathbf{x}| \notin S(k)$. If $n > f(k)k$, then $n \in R_{f(k)}(k)$ is of the form $n = f(k)k + v$ where $v = 1 \pmod{(f(k)+1)}$. If any of the components of $\mathbf{x}$ is one of

$$\{x : 1 \le x \le v \quad \text{and} \quad x \ne 0 \pmod{(f(k)+1)}\},$$

then removal of such a component reduces the norm to a value in $S(k)$. On the other hand, if $\mathbf{x}$ contains only components of sizes not in the above list, namely

$$\{x : 1 \le x \le v \quad \text{and} \quad x = 0 \pmod{(f(k)+1)}\} \cup \{x : x > v\},$$

then the smallest norm such an $\mathbf{x}$ can have occurs when $k-1$ of the components are $f(k)+1$ and the last component is at least $v+1$ since the norm of $\mathbf{x}$ cannot be a multiple of $f(k)+1$. That is $n \ge (k-1)(f(k)+1) + v + 1 = n + k - f(k) > n$. This contradiction shows that such $\mathbf{x}$ do not exist. The same argument works for $\mathbf{x}$ with norm $n \in R(k)$ satisfying $n < f(k)k$. Find $j < f(k)$ such that $jk < n < (j+1)k$; then $n = jk + v$

where $v = 1 \pmod{j+1}$. Then $\mathbf{x}$ can be moved directly to $S(k)$ if $\mathbf{x}$ has any component in $\{x : 1 \le x \le v \quad \text{and} \quad x \ne 0 \pmod{(j+1)}\}$ and otherwise its norm n is at least $(k-1)(j+1)+v+1 = n-j+k > n$. ■

Using this result, one may find all P-positions. A position, $\mathbf{x}$, with norm n is a P-position if and only if for every component x_i of $\mathbf{x}$ we have $n - x_i \in R(k)$. It is easy to see that for even values of k, the set of P-positions is the set of all positions with an odd number of chips in each box. For other values of k, the description is not so easy.

As an example, we give without proof the set of P-positions for $k = 7$. These positions can conveniently be grouped into four classes. There is an initial class, $\mathcal{P}_1$, of exceptions consisting of positions whose sum is less than 21, and then things settle down into one of three classes distinguished by the sum modulo 4. These three general classes are

$$
\begin{aligned}
\mathcal{P}_2 &= \{\mathbf{x} = (x_1, \ldots, x_7) : x_i = 3 \pmod 4 \text{ for all } i\} \\
\mathcal{P}_3 &= \{\mathbf{x} = 2^6 z : z = 0 \pmod 4 \text{ and } z \ge 8\} \\
\mathcal{P}_4 &= \{\mathbf{x} = (x_1, \ldots, x_7) : x_i = 1 \pmod 4 \text{ for all } i \text{ and } \textstyle\sum x_i \ge 23\} \\
&\qquad - \{1^5 9 z\} - \{1^4 5^2 z\}
\end{aligned}
$$

For example, the class $\mathcal{P}_4$ consists of all 7-tuples of positive integers equal to $1 \pmod 4$ whose sum is at least 23, except for those 7-tuples that contain five 1's and a 9, and those that contain four 1's and two 5's. The positions in class $\mathcal{P}_2$ have sum equal to $1 \pmod 4$, those in class $\mathcal{P}_3$ have sum equal to $0 \pmod 4$, and those in $\mathcal{P}_4$ have sum equal to $3 \pmod 4$.

The class $\mathcal{P}_1$ of exceptions may be classified by the sum of the components. They are:

sum				
7	1^7			
9	$1^6 3$			
11	$1^6 5$	$1^5 3^2$		
13	$1^6 7$	$1^5 3\ 5$	$1^4 3^3$	
14	2^7			
16	$1^6 10$	$1^4 4^3$		
17	$2^6 5$			
19	$1^6 13$	$1^5 7^2$	$1^4 4^2 7$	$1^3 4^4$
20	$2^5 5^2$			

The set of P-positions is the union of these four classes, $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4$.

The misère version of this game can also be solved. Unlike what usually turns out to be the case, the misère version is somewhat simpler than the normal version. The function corresponding to the function $f(k)$ above takes on only three values, allowing us to consider just three cases. We state the result separately for the three cases without proof.

Proposition 8(misère). *(1) If k is odd, the set of P-positions is $\mathcal{P} = \{1^{k-1} z : z \text{ even}\}$.*
(2) Suppose $k = 0 \pmod 6$ or $k = 2 \pmod 6$.

For n odd, $k < n < 2k$, let $\mathbf{x}(n) = 1^{k-1}z$.
For $n \geq 2k$, $n - 2k = 0 \pmod 3$, let $\mathbf{x}(n) = 2^{k-1}z$.
For $n \geq 2k$, $n - 2k = 2 \pmod 3$, let $\mathbf{x}(n) = 1^{k-1}z$.
All such $\mathbf{x}(n)$ are P-positions. For all other n, positions $\mathbf{x}$ with $|\mathbf{x}| = n$ are N-positions.
(3) Suppose $k = 4 \pmod 6$.
For n odd, $k < n < 2k$, let $\mathbf{x}(n) = 1^{k-1}z$.
For $2k \leq n < 3k$, $n - 2k = 0 \pmod 3$, let $\mathbf{x}(n) = 2^{k-1}z$.
For $2k \leq n < 3k$, $n - 2k = 2 \pmod 3$, let $\mathbf{x}(n) = 1^{k-1}z$.
For $n \geq 3k$, $n - 3k = 0 \pmod 4$, let $\mathbf{x}(n) = 3^{k-1}z$.
For $n \geq 3k$, $n - 3k = 3 \pmod 4$, let $\mathbf{x}(n) = 2^{k-1}z$.
For $n \geq 3k$, $n - 3k = 2 \pmod 4$, let $\mathbf{x}(n) = 1^{k-1}z$.
All such $\mathbf{x}(n)$ are P-positions. For all other n, positions $\mathbf{x}$ with $|\mathbf{x}| = n$ are N-positions.

As an example, suppose $k = 10$ and $\mathbf{x} = 3^5 4^2 7\ 11\ 29$. Then $n = |\mathbf{x}| = 70$. Since $70 \geq 3k = 30$, and $n - 3k = 0 \pmod 4$, we may not remove a component of size $3 \pmod 4$ because the resulting $n - 3k$ would be equal to $1 \pmod 4$. Thus the two optimal moves according to Proposition 8(misère) are: Empty a box of 4 chips and move to $3^9 39$, or empty the box of 29 chips and move to $2^9 23$.

5. Entropy Reduction.

Rules: There are k boxes of chips. A legal move consists of transferring chips from one box to another in such a way as to reduce the entropy. That is, a move is to take chips from one box and put them in another provided the boxes are made more nearly equal in size. That the game eventually ends follows by noticing that each move reduces the sum of squares or the sum of the absolute differences by at least 2. Last to move wins.

A position may be taken as a k-tuple of integers. Clearly the position is not changed if we add the same number of chips to each box. We may even allow initial positions with a negative number of chips in a box. The terminal positions are those in which the numbers of chips in all the boxes differ by at most 1. We restrict attention to the case of three boxes.

Proposition 9. *When $k = 3$, the P-positions are triplets of the form (x, x, y), where $x = y$ or where $|x - y|$ in its binary representation ends in an even number of zeros, or equivalently, where $x = y$ or $|x - y|$ may be written in the form $4^n m$ for some nonnegative integer n and some odd integer m.*

Proof. 1. The terminal positions are of the form (x, x, x) and $(x, x, x \pm 1)$. These are clearly of the proper form with $x = y$ or $|x - y| = 1$, and so are in $\mathcal{P}$.

2. Suppose $(x, x, y) \in \mathcal{P}$. If $x = y$, the position is terminal. Otherwise, the only move back to a position with two equal coordinates occurs by moving to make y and one of the x equal, namely to (x, u, u) where $u = (x + y)/2$. Then since $x \neq y$ and $|x - y| = 4^n m$ for m odd, we have $|x - u| = |x - y|/2$ which is not of the form 4^n times an odd number. Thus, $(x, u, u) \in \mathcal{N}$.

3. If $(x, x, y) \in \mathcal{N}$, then $|x - y| = 4^n 2m$ with m odd, and we can move (x, x, y) into (x, u, u) with $u = (x + y)/2$. But $|x - u| = |x - y|/2 = 4^n m$ and so $(x, u, u) \in \mathcal{P}$. So assume $(x, y, z) \in \mathcal{N}$ with $x > y > z$. We may move chips from the x box into the z box

until one of the boxes is of size y. If this moves (x,y,z) into (u,y,y) or (y,y,u) with $u=y$ or $|u-y|=4^n m$ with m odd, we are done since the resulting position is in $\mathcal{P}$. If not, that is if $|u-y|=4^n 2m$ with m odd, we continue moving chips from the first box into the third until we arrive at (v,y,v) where $v=(u+y)/2$. Then $|v-y|=|u-y|/2=4^n m$, so that $(v,y,v)\in\mathcal{P}$. ■

It is interesting to note that for a given $(x,y,z)\in\mathcal{N}$, there always exists a move to $\mathcal{P}$ by moving chips from the largest box into the smallest.

We can also solve the misère version of the game. The set of P-positions is more complex and it is advantageous to simplify the notation. The outcome of a position (x,y,z) is invariant under permutation of the pile sizes (i.e. changing to (y,z,x)), and under subtraction of a constant from each of the piles (i.e. changing to $(x-u,y-u,z-u)$. Therefore we may simplify the notation by writing (x,y,z) in nonincreasing order, normalizing so that the third coordinate is 0, and then dropping the third coordinate from the notation. Thus $(5,2)$ represents any position of the form $(u+5,u+2,u)$, $(u+2,u,u+5)$, etc. In addition the outcome is invariant under negation (i.e. changing (x,y,z) to $(u-x,u-y,u-z)$). Thus $(5,3)$ and $(5,2)$ must have the same outcome; similarly $(x,0)$ and (x,x) have the same outcome. Although we do not take advantage of this in the notation, taking note of it will simplify the proof.

Proposition 9(misère). *For the misère version of the entropy game with $k=3$, the P-positions are $\mathcal{P}=\mathcal{P}_1\cup\mathcal{P}_2\cup\mathcal{P}_3$, where*

$$\begin{aligned}
\mathcal{P}_1 &= \{(2,0),(2,1),(2,2),(5,2),(5,3)\}\\
\mathcal{P}_2 &= \{(x,0),(x,x) : x=4^n 2m \text{ for } m=3 \text{ or } 7\}\\
\mathcal{P}_3 &= \{(x,0),(x,x) : x=4^n m \text{ for } m \text{ odd, } m\neq 3 \text{ or } 7\text{, and when } n\leq 1,\ m\neq 1\}
\end{aligned}$$

Proof. 1. The terminal positions are $(0,0)$, $(1,0)$ and $(1,1)$ and are not in $\mathcal{P}$.

2. It is easy to check that the five elements of $\mathcal{P}_1$ cannot move to elements of $\mathcal{P}$. Now suppose $(x,0)\in\mathcal{P}_2$ with $x=4^n 2m$, $m=3$ or 7. The only moves are to $(x-u,u)$. This cannot be in $\mathcal{P}_1$ because the sum x is a multiple of 6 or 14. It can be in $\mathcal{P}_2\cup\mathcal{P}_3$ only if $x-u=u$; that is, only at $(x/2,x/2)$. But $x/2$ is of the form $4^n m$ with $m=3$ or 7 and so can be in neither $\mathcal{P}_2$ nor $\mathcal{P}_3$. By symmetry, $(x,x)\in\mathcal{P}_2$ cannot move to $\mathcal{P}$. Finally, suppose $(x,0)\in\mathcal{P}_3$ with $x=4^n m$, m odd. ($(x,x)\in\mathcal{P}_3$ may be treated by symmetry.) The only moves are to $(x-u,u)$. This cannot be in $\mathcal{P}_1$ because $m=3$ and $m=7$, as well as $n=1$, $m=1$ have been excluded from $\mathcal{P}_3$. It can be in $\mathcal{P}_2\cup\mathcal{P}_3$ only at $(x/2,x/2)$. But this can occur only if $n>0$ and x has the form $4^{n-1}2m$, m odd. This cannot be in $\mathcal{P}_3$, nor can it be in $\mathcal{P}_2$ since $m=3$ and $m=7$ have been excluded from $\mathcal{P}_3$.

3. Now suppose $(x,y)\notin\mathcal{P}$. We must show that if (x,y) is not terminal, it can be moved to $\mathcal{P}$. First suppose (x,y) is of the form $(x,0)$ (or symmetrically (x,x)). Then x must be equal to 1, 4, $4^n m$ for $n>1$ and $m=3$ or 7, or $4^n 2m$ for m odd $m\neq 3$ or 7. Take them in order. $(1,0)$ is terminal. $(4,0)$ can be moved to $(2,2)\in\mathcal{P}_1$. $(x,0)$ for $x=4^n m$ with $n>1$ and $m=3$ or 7 can be moved to $(x/2,x/2)\in\mathcal{P}_2$ since $x/2=4^{n-1}2m$ with $m=3$ or 7, which is in $\mathcal{P}_2$. $(x,0)$ for $x=4^n 2m$ with $m\neq 3$ or 7 can be moved to $(4^n m,4^n m)$, which is in $\mathcal{P}_3$ except when $m=1$ and $n=0$ or 1; in which case $(2,0)$ is initially in $\mathcal{P}_1$ and $(8,0)$ can be moved to $(5,3)$.

The remaining cases to be considered are all those (x,y) with $x > y > 0$, except for $(2,1)$, $(5,2)$ and $(5,3)$. We may without loss of generality assume $y \geq x/2$. There are two main cases. First suppose that x is odd. Then transferring $x-y$ chips from the first box to the third moves to the position $(2y-x, 2y-x)$. This is of the form $4^n m$ with $n=0$ m odd, and so is in $\mathcal{P}_3$ provided $m \neq 1$, $m \neq 3$ and $m \neq 7$. If $m=1$, the move was to $(1,1)$ and by transferring two chips less you could have moved to $(5,3) \in \mathcal{P}_1$, unless you started at $(x,y)=(3,2)$, in which case you could move to $(2,0) \in \mathcal{P}_1$. If $m=3$, the move was to $(3,3)$ and by transferring one more chip you could have moved to $(2,1) \in \mathcal{P}_1$. If $m=7$, the move was to $(7,7)$ and by transferring two more chips you could have moved to $(5,3) \in \mathcal{P}_1$.

Now suppose x is even. Transferring $x-y$ chips from the first box to the third moves to the position $(2y-x, 2y-x)$, and transferring $x/2$ chips from the first to the third leads to $(y-(x/2), 0)$. Since x is even, $2y-x$ is either of the form $4^n m$ with m odd and $n \geq 1$, or of the form $4^n 2m$ with m odd $n \geq 0$. If $y-2x = 4^n m$ with m odd and $n \geq 1$, then $(2y-x, 2y-x) \in \mathcal{P}_3$ unless $m=3$ or $m=7$, in which case $(y-(x/2), 0) \in \mathcal{P}_2$; or $m=1$ and $n=1$, in which case $(y-(x/2),0) = (2,0) \in \mathcal{P}_1$. Finally if $2y-x = 4^n 2m$ with m odd and $n \geq 0$, then $(2y-x, 2y-x) \in \mathcal{P}_2$ if $m=3$ or $m=7$; if $m \neq 3$ or 7, then $(y-(x/2),0) \in \mathcal{P}_3$ unless $m=1$ and $n=0$ or 1, that is $y-2x=2$ or 8. If $n=0$, then $(y-2x, y-2x) = (2,2) \in \mathcal{P}_1$, and if $n=1$ then $(2y-x, 2y-x) = (8,8)$, and transferring 3 more chips would end up at $(5,3) \in \mathcal{P}_1$. ■

The proofs are constructive. One can see that the number of arithmetic operations needed to find the outcome and an optimal move from a given N-position is linear in the logarithm of the maximum difference of the box sizes.

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