On a Rao-Shanbhag Characterization of the Exponential/Geometric Distribution

Thomas S. Ferguson, UCLA

Abstract: A theorem of Rao and Shanbhag, characterizing the exponential and the geometric distributions as the only distributions of independent X and Y for which $X - Y$ is independent of $\{\min\{X, Y\} \leq u\}$ for some suitable values of u, is discussed and a generalization to many variables is explored.

KEY WORDS: Characterization by independence, Lau-Rao Theorem.

1. Introduction. The main purpose of this paper is to present a generalization to many variables of a theorem of C. R. Rao and D. N. Shanbhag characterizing the exponential and geometric distributions on the basis of independence of certain statistics. A secondary purpose is to bring attention to an unpublished thesis of Chia-Jon Hong (1985) containing some results giving rise to this generalization. A related theorem of Srivastava is extended to the absolutely continuous case.

In the discussion, the exponential and geometric distributions are taken to be two and three parameter families including location and scale parameters. Thus, the exponential distributions have density

$$f(x|\alpha, \theta) = \theta^{-1} \exp\{-(x - \alpha)/\theta\} \mathrm{I}(x > \alpha) \tag{1}$$

where $\theta > 0$ and α are real numbers, and $\mathrm{I}(A)$ represents the indicator function of the set A. The geometric distributions have probability mass function

$$f(x|\alpha, \lambda, p) = (1 - p)p^n \quad \text{for } x = \alpha + n\lambda \text{ and } n = 0, 1, 2, \ldots \tag{2}$$

where $0 < p < 1$, $\lambda > 0$, and α are real numbers.

With this understanding, the underlying result that forms the basis of the Rao-Shanbhag characterization of the exponential and geometric is the following theorem due to Ferguson (1964, 1965) and Crawford (1966).

Theorem. *(Ferguson and Crawford) Let X and Y be independent nondegenerate random variables, and let $U = \min\{X, Y\}$ and $V = X - Y$. Then U and V are independent if and only if X and Y are both exponential or both geometric with the same support.*

Theorem 8.2.1 in the book of Rao and Shanbhag (1994) improves this characterization by weakening the independence assumption as follows.

Theorem. *(Rao and Shanbhag) Let X and Y be independent random variables, and let $U = \min\{X, Y\}$ and $V = X - Y$. Suppose u_1 is any number such that there are at least two points of the support of the distribution of U in $(-\infty, u_1]$. Then V is independent of* *the events $\{U \leq u\}$ for all $u \leq u_1$ if and only if X and Y are both exponential or both geometric with the same support.*

Thus, complete independence between U and V is not needed. All that is required is that V be independent of the events $\{U \leq u\}$ for a "few" values of u (at least two) in the left tail of the distribution of U. The proof of this theorem uses the celebrated Lau-Rao Theorem (1982). In fact, the main theme of their 1994 book is the power and unity that the Lau-Rao Theorem brings to many areas.

It should be noted that we have replaced the condition that V and $U\mathrm{I}(U \leq u_1)$ be independent as given by Rao and Shanbhag in the statement of their theorem by the stronger condition that V and the events $\{U \leq u\}$ be independent for all $u \leq u_1$. This is the condition I am sure they intended and the one that was used in the proof. Hong (1985) gives an example of independent identically distributed random variables X and Y, whose common distribution is an equiprobable mixture of two distinct geometric distributions on the nonnegative integers, for which V and $\{U = 1\}$ are independent. This gives a counterexample to the theorem of Rao and Shanbhag as stated in their book. It also answers a question raised by Arnold (1980) in a related context.

There are many related results characterizing the exponential and geometric distributions in the statistical literature. Phillips (1981) has treated the Ferguson/Crawford problem in which independence of U and V is replaced by the conditional independence of U and V given $V > 0$. An extension of this to many variables is given in Hong (1985) and Liang and Balakrishnan (1992).

Another class of problems stems from the work of Fisz (1958), Rogers (1963) and Ferguson (1967). In this class, the variables $X_1, X_2, \ldots, X_n$ are assumed to be independent and identically distributed, and distributions are characterized by the linearity of the regression of one order statistic on another. The distributions so characterized include the power distributions and the Pareto distributions as well as the exponential and geometric distributions. Work on various aspects of this problem was done by El-Neweihi and Govindarajulu (1979), Sreehari (1983), Rao and Shanbhag (1986), Wu and Ouyang (1996) and others, but the original problem was not solved in full generality until Dembinski and Wesolowski (1998).

Another class of problems involves characterizing distributions by equality of distribution of certain statistics. Interesting papers in this area include Puri and Rubin (1970), Fosam and Shanbhag (1994) and Rao and Shanbhag (1996).

2. Extension to Many Variables.

The unpublished thesis of Chia-Jon Hong (1985) contains generalizations to many variables of special cases of this result. One of Hong's results for the geometric distribution is as follows.

Theorem. *(Hong) Let $X_1, \ldots, X_n$ be independent nonnegative integer-valued random variables and suppose that $\mathrm{P}(X_j = 0) > 0$ and $\mathrm{P}(X_j = 1) > 0$ for some j, $1 \leq j \leq n$.*

Let $U = \min\{X_1, \ldots, X_n\}$. If the event $\{U = 0\}$ and the vector of differences, $\boldsymbol{V} = (X_2 - X_1, \ldots, X_n - X_1)$, are independent, then all $X_1, \ldots, X_n$ have geometric distributions with support $\mathbb{N}$.

This may be considered as a generalization to many variables of a discrete version of the theorem of Rao and Shanbhag, where the support of the distributions is known to be restricted to the nonnegative integers. However even in the case $n = 2$, this does not follow from the theorem of Rao and Shanbhag because independence of V and $\{U = u\}$ is required for only one value of u.

Hong's corresponding result for the exponential distribution is

Theorem. *(Hong) Let $X_1, \ldots, X_n$ be independent nonnegative random variables with continuous distributions. Suppose 0 is in the support of the distribution of X_1, say. If for a sequence of $\epsilon > 0$ tending to zero, the event $\{U < \epsilon\}$ and $\boldsymbol{V}$ are independent, then all $X_1, \ldots, X_n$ are exponential with support $(0, \infty)$.*

This is an extension to many dimensions of a continuous version of the theorem of Rao and Shanbhag. Hong proved the first theorem using the Shanbhag Lemma (1977). For the second theorem, he used a theorem of Ramachanran (1977) and Shimizu (1978). He was not aware of the Lau-Rao Theorem.

But now that we have the example of Rao and Shanbhag to guide us, we can unify and generalize Hong's results. This requires the following extension of the Lau-Rao Theorem.

Theorem 1. *Let μ be a σ-finite measure on $[0, \infty)$ satisfying the condition $\mu(\{0\}) < 1$. Let $f_1(x), \ldots, f_n(x)$ be nonnegative real-valued Borel measurable functions defined on $[0, \infty)$, locally integrable with respect to Lebesgue measure, none of which are identically zero a.e. (almost everywhere with respect to Lebesgue measure). Suppose*

$$\prod_{i=1}^{n} f_i(x_i) = \int_0^\infty \prod_{i=1}^{n} f_i(x_i + y)\, d\mu(y) \quad \text{a.e.} \tag{4}$$

Then either μ is arithmetic with some span $\lambda > 0$ and

$$f_i(x + k\lambda) = f_i(x) b_i^k \quad \text{a.e. for } i = 1, \ldots, n \text{ and } k = 0, 1, \ldots, \tag{5}$$

where the $b_i > 0$ satisfy $\sum_{k=0}^{\infty} (\prod_{i=1}^{n} b_i)^k \mu(\{k\lambda\}) = 1$, or μ is nonarithmetic and

$$f_i(x) = c_i \exp\{\theta_i x\} \quad \text{a.e. for } i = 1, \ldots, n. \tag{6}$$

for some $c_i > 0$ and θ_i satisfying $\int_0^\infty \exp\{x \sum_1^n \theta_i\}\, d\mu(x) = 1$.

When $n = 1$ this is the Lau-Rao Theorem. This should be considered an extension rather than a generalization because it follows easily from the Lau-Rao Theorem itself. Based on this theorem, we can prove the following unification of the theorems of Hong.

Theorem 2. *Let $X_1, \ldots, X_n$ be independent. Assume that $U = \min\{X_1, \ldots, X_n\}$ is nondegenerate and let u_1 be any number such that $\mathrm{P}(U \le u_1) > 0$. If the the vector of differences, $\mathbf{V} = (X_2 - X_1, \ldots, X_n - X_1)$, is independent of the events $\{U \le u\}$ for $u \le u_1$, then $X_1, \ldots, X_n$ are all exponential with the same support or all geometric with the same support.*

This also contains the theorem of Rao and Shanbhag when $n = 2$, and it makes the minor improvement of not requiring two points of the support of U below u_1 in the discrete case. Proofs are deferred to Section 4.

3. A Theorem of Srivastava.

Another weakening of the independence assumption in the Theorem of Ferguson and Crawford occurs in a result of R. C. Srivastava (1974). Here the weakening is on the values of V on which independence is required but the setting is restricted to be discrete. Let $\mathbb{N}$ denote the natural numbers $\{0, 1, 2, \ldots\}$.

Theorem. *(Srivastava) Let X and Y be independent nondegenerate $\mathbb{N}$-valued random variables, and suppose the support of Y is $\mathbb{N}$. Let $U = \min\{X, Y\}$ and $V = X - Y$. Then U and $\{V = v\}$ are independent for $v = 0$ and $v = 1$ if and only if X and Y are both geometric with support $\mathbb{N}$.*

Here is an analogous theorem in the absolutely continuous case to characterize the exponential distribution.

Theorem 3. *Let X and Y be independent nonnegative absolutely continuous random variables and suppose that the support of Y is $[0, \infty)$. Let $U = \min\{X, Y\}$ and $V = X - Y$. Then U and $V\mathrm{I}(0 < V < \epsilon)$ are independent for some $\epsilon > 0$ if and only if X and Y are both exponential on $(0, \infty)$.*

This result is not true if the support of Y is not required to be the whole of $[0, \infty)$. One can find other distributions if the support of Y is taken to be $[0, u_0]$ for some $u_0 > 0$. The class of all such distributions is found in the following theorem in which we have taken $u_0 = 1$ since this is just a change of scale.

Theorem 4. *Let X and Y be independent nonnegative absolutely continuous random variables and suppose that the support of Y is $[0, 1]$. Let $U = \min\{X, Y\}$ and $V = X - Y$ and let ϵ be any positive number for which $\mathrm{P}(V < \epsilon) > 0$. Then U and $V\mathrm{I}(0 < V < \epsilon)$ are independent if and only if for some $\lambda > 0$ and some $\theta > 0$, Y has survival function*

$$\mathrm{P}(Y > u) = \left(\frac{e^{-\lambda u} - e^{-\lambda}}{1 - e^{-\lambda}} \right)^{\theta} \qquad \text{for } 0 < u < 1 \tag{7}$$

and X has density

$$f_X(x) = \frac{\theta \lambda}{\theta + e^{\lambda}} e^{-\lambda x} \qquad \text{for } 0 < x < 1 + \epsilon. \tag{8}$$

($f_X(x)$ is arbitrary for $x > 1 + \epsilon$.)

4. Proofs.

Proof of Theorem 1. There exist, for $i = 1\dots,n$, x_i^0 with $f_i(x_i^0) > 0$ and such that the function $h(x) = \prod_{i=1}^n f_i(x_i^0 + x)$ satisfies the conditions of the Lau-Rao Theorem.

Case 1. μ is nonarithmetic: Then $h(x) = c\exp\{\theta x\}$ a.e. for some $c > 0$ and θ satisfying $\int_0^\infty e^{\theta x}\, d\mu(x) = 1$. Thus $f_i(x) > 0$ a.e. for $x \geq x_i^0$. Then (4) implies $f_i(x) > 0$ a.e. for all $x \geq 0$. Now fix j and apply the Lau-Rao Theorem once more to (4) with $x_i = 0$ for $i \neq j$. This implies that $f_j(x) = c_j \exp\{\theta_j x\}$.

Case 2. μ is arithmetic with span λ: Then $h(x + k\lambda) = h(x)b^k$ for $k = 0, 1, \dots$ and $x \geq 0$, where $b > 0$ satisfies $\sum_0^\infty b^k \mu(\{k\lambda\}) = 1$. This implies there are $x_i^1 \in [0, \lambda)$ such that $f_i(x_i^1) > 0$. Now fix j and apply the Lau-Rao Theorem once more to (1) with $x_i = x_i^1$ for $i \neq j$. This implies $f_i(x + k\lambda) = f_i(x)b_i^k$ for $k = 0, 1, \dots$ and $x \geq 0$. ■

Proof of Theorem 2. Note that $\{U = X_i\}$ is a function of $\boldsymbol{V}$ for all $i = 1, \dots, n$. Then the independence of $\boldsymbol{V}$ and $\{U \leq u\}$ implies that $\boldsymbol{V}$ and $\{U \leq u\}$ are conditionally independent given $\{U = X_i\}$. This is because

$$\begin{aligned} \mathrm{P}(U \leq u, \boldsymbol{V} \leq \boldsymbol{v} | U = X_i) &= \mathrm{P}(U \leq u, \boldsymbol{V} \leq \boldsymbol{v}, U = X_i)/\mathrm{P}(U = X_i) \\ &= \mathrm{P}(U \leq u)\mathrm{P}(\boldsymbol{V} \leq \boldsymbol{v}, U = X_i)/\mathrm{P}(U = X_i) \\ &= \mathrm{P}(U \leq u | U = X_i)\mathrm{P}(\boldsymbol{V} \leq \boldsymbol{v} | U = X_i). \end{aligned}$$

The second equality follows because $\{U \leq u\}$ is independent of $\mathbf{V}$ and $\{U = X_i\}$ is a function of $\boldsymbol{V}$. The third equality follows similarly.

Let u_0 denote the left extremity of the support of U (possibly $-\infty$). Then for some i, u_0 is the left extremity of the support of X_i. Suppose without loss of generality that u_0 is the left extremity of the support of X_1. Then

$$\mathrm{P}(U \leq u, U = X_1) > 0 \tag{9}$$

for all $u > u_0$. Therefore, for all $\boldsymbol{v} = (v_2, \dots, v_n) \geq \boldsymbol{0}$,

$$\mathrm{P}(\boldsymbol{V} \geq \boldsymbol{v} | U \leq u, U = X_1) = \mathrm{P}(\boldsymbol{V} \geq \boldsymbol{v} | U = X_1), \tag{10}$$

independent of u for $u_0 < u \leq u_1$. If $u_0 = -\infty$, then letting $u \to -\infty$ in (10) shows that (10) must be 1 for all $\boldsymbol{v} \geq \boldsymbol{0}$. But letting $\boldsymbol{v} \to \infty$ in (10) gives the value 0 which is a contradiction. Thus, u_0 must be finite. Without loss of generality, take $u_0 = 0$.

We distinguish two cases. If U has a mass point at $u_0 = 0$, we may assume without loss of generality that X_1 has a mass point at 0. Then for all $\boldsymbol{v} > \boldsymbol{0}$,

$$\begin{aligned} \mathrm{P}(\boldsymbol{V} \geq \boldsymbol{v})\mathrm{P}(U = 0) &= \mathrm{P}(\boldsymbol{V} \geq \boldsymbol{v}, U = 0) \\ &= \mathrm{P}(X_1 = 0, X_2 \geq v_2, \dots, X_n \geq v_n) \\ &= \mathrm{P}(X_1 = 0)\mathrm{P}(X_2 \geq v_2) \cdots \mathrm{P}(X_n \geq v_n) \end{aligned} \tag{11}$$

Checking separately, we see that (11) must hold even if some of the v_i are zero; thus it holds for all $\boldsymbol{v} \geq \mathbf{0}$. Note that X_1 cannot be degenerate at 0 otherwise U would be also. We may now apply the extended Lau-Rao Theorem to conclude that either the distribution of X_1 is arithmetic with some span $\lambda > 0$, in which case,

$$\mathrm{P}(X_i \geq x + k\lambda) = \mathrm{P}(X_i \geq x) b_i^k \quad \text{for } i = 2, \ldots, n \text{ and } k = 0, 1, \ldots, \tag{12}$$

for some $0 < b_i < 1$, or the distribution of X_1 is nonarithmetic in which case, $\mathrm{P}(X_i \geq x) = c_i e^{\theta_i x}$ for $i = 2, \ldots, n$.

If U does not have a mass point at 0, then $\mathrm{P}(X_1 = U, U \leq u) > 0$ for all $u > 0$. Then from (10), we have for all $\boldsymbol{v} \geq \mathbf{0}$,

$$\mathrm{P}(\boldsymbol{V} \geq \boldsymbol{v} | U = X_1) = \lim_{u \to 0+} \mathrm{P}(\boldsymbol{V} \geq \boldsymbol{v} | U \leq u, U = X_1). \tag{13}$$

First note that for $v \geq 0$,

$$\mathrm{P}(\boldsymbol{V} \geq \boldsymbol{v} | U = X_1) = \frac{\mathrm{P}(\boldsymbol{V} \geq \boldsymbol{v}, U = X_1)}{\mathrm{P}(U = X_1)} = \frac{\mathrm{P}(\boldsymbol{V} \geq \boldsymbol{v})}{\mathrm{P}(U = X_1)}.$$

Similarly for the right side of (13):

$$\begin{aligned} \mathrm{P}(\boldsymbol{V} \geq \boldsymbol{v} | U \leq u, U = X_1) &= \mathrm{P}(\boldsymbol{V} \geq \boldsymbol{v} | X_1 \leq u, U = X_1) \\ &= \frac{\mathrm{P}(\boldsymbol{V} \geq \boldsymbol{v}, U = X_1 | X_1 \leq u)}{\mathrm{P}(U = X_1 | X - 1 \leq u)} \\ &= \frac{\mathrm{P}(\boldsymbol{V} \geq \boldsymbol{v} | X_1 \leq u)}{\mathrm{P}(U = X_1 | X - 1 \leq u)} \to \mathrm{P}(X_2 > v_2, \ldots, X_n > v_n) \end{aligned}$$

as $u \to 0+$. Therefore (13) becomes

$$\mathrm{P}(X_2 \geq X_1 + v_2, \ldots, X_n \geq X_1 + v_n) = \mathrm{P}(U = X_1)\mathrm{P}(X_2 > v_2, \ldots, X_n > v_n)$$

The left side is left-continuous in v and the right side is right-continuous. This shows that the distributions of $X_2, \ldots, X_n$ are continuous, and so $X_j > v_j$ in (12) may be replaced by $X_j \geq v_j$. Since the distribution of X_1 is nonarithmetic in this case, the Lau-Rao Theorem implies that $X_2, \ldots, X_n$ have exponential distributions on $(0, \infty)$. ■

Proof of Theorem 3. The "if" part is well known, We prove the converse. For all $v \geq 0$ and all $u \geq 0$,

$$f_{U,V}(u, v) = f_{X,Y}(u + v, u) = f_X(u + v) f_Y(u). \tag{14}$$

The first equality follows because if $v \geq 0$, then $X - Y = v$ implies $X \geq Y$, so that $U = Y$. For $0 \leq v \leq \epsilon$ and all $u \geq 0$, we have the Cauchy functional equation,

$$f_U(u) f_V(v) = f_X(u + v) f_Y(u). \tag{15}$$

Since we have assumed $f_Y(u) > 0$ for almost all $u \geq 0$, this implies that $f_X(x) = \lambda e^{-\lambda x}$ for some $\lambda > 0$. The density of U is

$$f_U(u) = f_X(u)\mathrm{P}(Y > u) + f_Y(u)\mathrm{P}(X > u) \\ = \lambda e^{-\lambda u}\mathrm{P}(Y > u) + f_Y(u)e^{-\lambda u}, \tag{16}$$

so that (15) implies

$$\lambda e^{-\lambda(u+v)} f_Y(u) = e^{-\lambda u}[\lambda \mathrm{P}(Y > u) + f_Y(u)] f_V(v)$$

From this, we see that $f_V(v)/e^{-\lambda v}$ is constant for $0 \leq v \leq \epsilon$. This in turn implies that $f_Y(u)/\mathrm{P}(Y > u)$ is constant for all $u > 0$. Thus Y is exponential also. ∎

Proof of Theorem 4. Since the support of Y is $[0, 1]$, (15) holds for all $0 < u < 1$ and all $0 < v < \epsilon$. This implies

$$f_X(x) = \mu e^{-\lambda x} \qquad \text{for } 0 < x < 1 + \epsilon \tag{17}$$

for some $\mu > 0$ and some λ. The distribution of X above $1 + \epsilon$ is not restricted. However, we cannot have mass greater than one on $(0, 1 + \epsilon)$, so

$$\begin{cases} 0 < \mu \leq \lambda/(1 - e^{-\lambda(1+\epsilon)}) & \text{for } \lambda \neq 0 \\ 0 < \mu \leq 1/(1 + \epsilon) & \text{for } \lambda = 0 \end{cases} \tag{18}$$

Now (15) becomes

$$f_U(u) f_V(v) = \mu e^{-\lambda(u+v)} f_Y(u) \qquad \text{for } 0 < u < 1 \text{ and } 0 < v < \epsilon. \tag{19}$$

This implies that $f_V(v) = \mu e^{-\lambda v}$ times some constant for $0 < v < \epsilon$. Substituting this into (19) gives

$$f_U(u) = e^{-\lambda u} f_Y(u)/c \qquad \text{for } 0 < u < 1, \tag{20}$$

where the constant c must be $\int_0^1 e^{-\lambda y} dF_Y(y)$. Since $U = \min\{X, Y\}$, $F_U(u)$ must be strictly greater than $F_Y(u)$ for all u. But from (20) this cannot be for u close to 0 unless $c < 1$. Thus $\lambda > 0$. The density of U is

$$f_U(u) = \mu e^{-\lambda u}\mathrm{P}(Y > u) + f_Y(u)(1 - \frac{\mu}{\lambda}(1 - e^{-\lambda u})). \tag{21}$$

Combining (20) and (21) gives

$$f_Y(u)[(\frac{1}{c} - \frac{\mu}{\lambda})e^{-\lambda u} - (1 - \frac{\mu}{\lambda})] = \mu e^{-\lambda u}\mathrm{P}(Y > u) \tag{22}$$

for $0 < u < 1$. The term in square brackets must be positive for all $0 < u < 1$, so assuming $(1/c) - (\mu/\lambda) \neq 0$ and integrating $f_Y(u)/\mathrm{P}(Y > u)$ gives

$$-\log \mathrm{P}(Y > u) = \mu \int \frac{e^{-\lambda u}}{(\frac{1}{c} - \frac{\mu}{\lambda})e^{-\lambda u} - (1 - \frac{\mu}{\lambda})} du = -\theta \log((\frac{1}{c} - \frac{\mu}{\lambda})e^{-\lambda u} - (1 - \frac{\mu}{\lambda}))$$

plus a constant of integration, where

$$\theta = \frac{\mu/\lambda}{\frac{1}{c} - \frac{\mu}{\lambda}} = \frac{\mu c}{(\lambda - \mu c)}. \tag{23}$$

But $\lim_{u \to 1-} \mathrm{P}(Y > u) = 0$ shows that we must have $\theta > 0$ and

$$(1 - \frac{\mu}{\lambda}) = (\frac{1}{c} - \frac{\mu}{\lambda})e^{-\lambda} \tag{24}$$

A similar argument shows that $(1/c) - (\mu/\lambda)$ cannot be equal to 0. Now, $\mathrm{P}(Y > 0) = 1$ fixes the constant of integration and Equation (7) follows. Solving (23) for c and substituting into (24) gives $\mu = \theta\lambda/(\theta + e^{\lambda})$. Equation (17) becomes Equation (8). It is straightforward to check the converse. ■

References.

Arnold, B. C. (1980) Two characterizations of the geometric distribution, J. Appl. Prob. **17**, 570-573.

Crawford, G. B. (1966) Characterization of geometric and exponential distributions, *Ann. Math. Stat.* **37**, 1790-1795.

Dembinska, A. and Wesolowski, J. (1997) Linearity of regression for non-adjacent order statistics, *Metrika* **48**, 215-222.

El-Neweihi, E. and Govindarajulu, Z. (1979) Characterizations of geometric distributions and discrete IFR(DFR) distributions using order statistics, *Statist. Planning & Inf.* **3**, 85-90.

Ferguson, T. S. (1964) A characterization of the exponential distribution, *Ann. Math. Stat.* **35**, 1199-1207.

Ferguson, T. S. (1965) A characterization of the geometric distribution, *Amer. Math. Mo.* **71**, 256-260.

Ferguson, T. S. (1967) On characterizing distributions by properties of order statistics, *Sankhya A* **20**, 265-278.

Fisz, M. (1958) Characterization of some probability distributions, *Skandinavisk Aktuarietidskrift* **41**, 65-67.

Fosam, E. B. and Shanbhag, D. N. (1994) Certain characterizations of exponential and geometric distributions, *J. Royal Statist. Soc. B* **56**, 157-160.

Hong, Chia-Jon (1985) On characterizations of some distributions, Ph.D. dissertation, Mathematics Department, UCLA.

Lau, K. S. and C. R. Rao (1982) Integrated Cauchy functional equation and characterizations of the exponential law, *Sankhya A,* **44**, 72-90.

Liang, T.C. and N. Balakrishnan (1992) A characterization of exponential distributions through conditional independence, *J. Royal Statist. Soc. B* **54**, 269-271.

Phillips, M. J. (1981) A characterization of the negative exponential distribution with application to reliability theory, *J. Appl. Prob.* **18**, 652-659.

Puri, P. S. and Rubin, H. (1970) A characterization based on the absolute difference of two i.i.d. random variables, *Ann. Math. Statist.* **41**, 2113-2122.

B. Ramachandran (1977) On the strong Markov property of the exponential laws, Paper contributed to the Colloquium on the Methods of Complex Analysis in the Theory of Probability and Statistics, held in Debrecen, Hungary in August -September 1977.

Rao, C. R. and Shanbhag, D. N. (1986) Recent results on characterization of probability distributions: A unified approach through extensions of Deny's theorem, *Advances Appl. Prob.* **18**, 660-678.

Rao, C. R. and Shanbhag, D. N. (1994) *Choquet-Deny type functional equations with applications to stochastic models*, Wiley, New York.

Rao, C. R. and Shanbhag, D. N. (1996) A note on a characteristic property based on order statistics, *Proc. Amer. Math. Soc.* **124**, 299-302.

Rogers, G. S. (1963) An altrnate proof of the characterization of the density Ax^B, *Amer. Math. Monthly* **70**, 857-858.

Shanbhag, D. N. (1977) An extension of the Rao-Rubin characterization of the Poisson distribution, *J. Appl. Prob.* **14**, 640-646.

R. Shimizu (1978) Solution to a functional equation and its application to some characterization problems, *Sankhya A*, **40**, 319-332.

Srivastava, R. C. (1974) Two characterizations of the geometric distribution, *J.Amer. Statist. Assoc.* **69**, 267-269.

Wu, J-W. and Ouyang, L. Y. (1996) On characterizing distributions by conditional expectations of functions of order statistics, *Metrica* **43**, 135-147.