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Misère Annihilation Games*

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Graph games with an annihilation rule, as introduced by Conway, Fraenkel and Yesha, are studied under the misère play rule for progressively finite graphs that satisfy a condition on the reversibility of non-terminal Sprague–Grundy zeros to Sprague–Grundy ones. Two general theorems on the Sprague–Grundy zeros and ones are given, followed by two theorems characterizing the set of P-positions under certain additional conditions. Application is made to solving many subtraction games, and solutions to two games not covered by the general theory are presented indicating a direction for future research. © 1984 Academic Press, Inc.

1. INTRODUCTION

This paper is concerned with games that are played on a directed graph, G, in the following manner. At the start of the game, a finite number of counters is placed on the vertices of the graph. A move consists of taking exactly one counter and moving it from its vertex along one of the directed edges to a new vertex. More than one counter may be at a vertex. Players alternate moves, and play continues until one of the players is unable to make a move because all counters are in terminal vertices of the graph, i.e., vertices from which no move is possible. If play continues forever, the game is declared a tie. If play stops, the player who moved last is declared the winner if the *normal play* rule is in force. Under the *misère play* rule, the player who moved last loses.

Games of this sort in which the graph may have circuits or loops are called "loopy games" and have been studied by Fraenkel and Perl [1], Fraenkel and Tassa [2], and Conway [3]. Some of the theory also appears in Chapter 11 of Conway's book [4]. Such games are *impartial* in the sense that the moves available at any position do not depend on which player is to move. *Partizan* loopy graph games in which there are two graphs on the

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same set of vertices, one graph for each player, have been studied by Flanigan [5].

In this paper, we restrict attention to graphs that are *progressively finite*, that is, for every vertex v of G the game starting with a single counter on v must end in a finite number of moves. Games on such a graph are not loopy; there are no ties. In addition, we assume that the Sprague–Grundy function (SG-function) of the graph is finite. The SG-function, g, is defined on the vertices of the graph with values in the non-negative integers as follows. Call a vertex w, a follower of vertex v, if there is a directed edge from v to w, denoted by $v \to w$. Then

$$\begin{aligned} g(v) &= \min\{m \in \mathbb{Z}: m \neq g(w) \text{ for some follower } w \text{ of } v\} \\ &= \operatorname{mex}\{g(w): v \to w\} \end{aligned} \tag{1}$$

where *mex* stands for minimal excludant (see Conway [4]), For terminal vertices v, $g(v) = 0$ by this definition. Then this definition may be applied inductively to determine the SG-values of other vertices.

A position in which there are n counters on vertices $v_1, ..., v_n$, duplication allowed, may be described by the multi-set $x = \{v_1, ..., v_n\}$. A *P-position* is one in which the *previous* player, the one who has just moved, can insure himself a win with optimal play. An *N-position* is one in which the player *next* to move can force a win with optimal play. Each position is either a P-position or an N-position. A game may be said to be solved if a simple description is given of the P-positions, one such that for a given position it can be determined in "polynomial time" whether or not the position is a P-position.

As an example, the well-known game Nim has a graph G whose vertices are the non-negative integers and whose directed edges are from any integer to any lesser integer. Thus, in the graph game Nim, any counter may be moved from its vertex, n, to any vertex m provided $m < n$. The only terminal vertex is 0, and it is easy to see that the SG-function takes values $g(n) = n$ for $n = 0, 1, 2, ...$.

The solution to this game was given by Bouton [6] in terms of the operation of binary addition of integers without carry, known as *nim-sum*. If r and s are non-negative integers with binary expansions $r = \sum_0^m r_i 2^i$ and $s = \sum_0^m s_i 2^i$ for some m, where each r_i and s_i are 0 or 1, then the nim-sum of r and s is $t = r \hat{+} s$, where $t = \sum_0^m t_i 2^i$ and $t_i = r_i + s_i \bmod 2$ and $t_i = 0$ or 1. Nim–sum is associative and commutative since addition mod 2 is. Bouton's characterization of the P-positions for Nim with the normal play rule has been generalized into the basic theorem of the subject of impartial games with normal play by Sprague [7] and Grundy [8]. This theorem states that for progressively finite graphs with finite SG-function g, a position

$x = \{v_1, \ldots, v_n\}$ is a P-position for normal play if, and only if, the nim-sum of the SG-values of the components is zero, that is,

$$g(x) \equiv g(v_1) \hat{+} \cdots \hat{+} g(v_n) = 0. \tag{2}$$

For misère play, the situation is much more complex. An excellent treatment of impartial games with misère play is found in Chapter 12 of Conway [4], in which one class of games called *tame* games are defined for which the solution can be easily found. A subclass of the tame games are the misère graph games whose graph satisfies the following condition.

CONDITION A. *Every non-terminal vertex of SG-value zero has a follower of SG-value one.*

It is easily seen that the game Nim satisfies this condition vacuously. Bouton [6] also solved the misère version of Nim and noted that it required only a small modification of the solution of the normal version of Nim. In Ferguson [9, Corollary of Theorem 2], it is noted that the natural generalization of Bouton's solution is also a solution to a general misère graph game if, and only if, the graph satisfies Condition A. (Note in that paper, a different definition of graph game is used, one that allows "splitting.") This solution is the following. Let

$$Q_1 = \{x = \{v_1, \ldots, v_n\} \mid g(x) = 1 \text{ and } g(v_i) \leqslant 1 \text{ for all } i\},$$

and

$$Q_0 = \{x = \{v_1, \ldots, v_n\} \mid g(x) = 0 \text{ and } g(v_i) > 1 \text{ for some } i\}.$$

Then the set of P-positions is

$$P = Q_1 \cup Q_0. \tag{3}$$

The concept of *annihilation games* was proposed by Conway and studied by Fraenkel [10] and Fraenkel and Yesha [11]. See also Fraenkel, Tassa and Yesha (1978) [12]. Such games are played on a directed graph with a finite number of counters on *distinct* vertices of the graph. Play proceeds as in ordinary graph games until some counter is moved to a new vertex which is already occupied by another counter, when the *annihilation rule* comes into play: both counters are annihilated, i.e., removed from play. The multiset x used to describe a position now becomes a set.

For progressively finite graphs with finite SG-functions, the P-positions for normal play in annihilation games are exactly the same as those in games without the annihilation rule, namely, positions that satisfy (2). For loopy graphs, however, annihilation games are quite distinct and it is to these games that Fraenkel and Yesha devoted their study.

It is the purpose of this paper to study annihilation games on progressively finite graphs with finite SG-functions under the misère play rule. It turns out that the theory is quite distinct from and more complex than that without the annihilation rule. Even misère annihilation Nim is distinct from misère Nim. For immediate comparison we give the solution here; this follows immediately from Theorem 3 in Section 3.

For misère annihilation Nim, the set of P-positions is $P = Q_1 \cup Q_0$, where

$$Q_1 = \{x = \{v_1, \ldots, v_n\} \mid g(x) = 1 \text{ and the } v_i \geqslant 2 \text{ can be grouped into pairs of the form } \{2j, 2j+1\} \text{ for various } j \geqslant 1\},$$

$$Q_0 = \{x = \{v_1, \ldots, v_n\} \mid g(x) = 0 \text{ and the } v_i \geqslant 2 \text{ cannot be so grouped}\}.$$

As an example, if $x = \{10, 21, 20, 11\}$, then $g(x) = 10 \hat{+} 11 \hat{+} 20 \hat{+} 21 = 0$ but x is not in Q_0 since there exists a grouping into pairs $\{10, 11\}$ and $\{20, 21\}$. This can be moved into Q_1 by using one of the annihilation moves, $21 \to 20$ or $11 \to 10$.

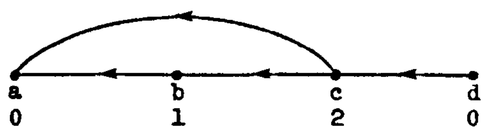
In Section 2, it is shown for misère annihilation games on progressively finite graphs with finite SG-function satisfying Condition A, that there is only one kind of vertex of SG-value 0 and only one kind of vertex of SG-value 1. That is, a counter on one vertex of SG-value 0 or 1 may be transferred to another vertex of the same SG-value without changing the outcome of the game. That this is not true for vertices of SG-value 2 or greater may be seen by referring to the examples of Section 4.

In Section 3, two general theorems are given that characterize the set of P-positions in an easily computable fashion, provided certain additional conditions are satisfied. The first theorem applies to unbounded SG-functions provided the graph satisfies an additional Condition B on the edges joining vertices of SG-values greater than one. In the special case of graphs with SG-functions having values no greater than 3, a considerably weaker Condition C is used in the next theorem.

In Section 4, the theorems of Section 3 are applied to the analysis of various subtraction games. Such games are known to satisfy Condition A (Ferguson [9]). In particular, all subtraction games with subtraction sets a subset of $\{1, 2, 3, 4, 5, 6, 7\}$ are scanned to see how successful the theorems of Section 3 are. On this basis, one feels they are quite successful, and the variety of simpler statements one can give to the solutions show the strength and usefulness of the theorems. The failures are collected in Sections 4.10 and 4.11. The former section indicates a need to generalize Theorem 3, as none of the games there have been solved. The latter section contains the only two games with SG-values no greater than three found earlier that did not satisfy Condition C. Solutions to both games are given in this section indicating that Condition C can be weakened, and holding out hope that a general solution to games with SG-values no greater than 3, satisfying Condition A, can be found.

2. Uniqueness of SG-0s and SG-1s

It is assumed throughout the rest of this paper that G is a progressively finite graph with finite SG-function, g. *It is also assumed throughout without further explicit mention that Condition A is satisfied.* This is a fairly restrictive assumption. For example, it rules out the very simple graph



The vertex d has SG-value 0 and its only follower is c of SG-value 2. No general theorems, not using Condition A, are presented here.

We are concerned solely with annihilation games on G under the misère play rule. Without the annihilation rule, the P-positions for all such games are described by (3), the analogue of Bouton's solution. Henceforth, the word "game" will be short for misère annihilation game. In addition, we use the notation *SG-k* as short for *vertex of SG-value k*, where k is a non-negative integer.

The first theorem implies that all SG-0s are equivalent in the sense that transferring a counter from one SG-0 to another does not change the outcome (i.e., a P-position stays P, and an N-position stays N). In particular, removing the counter from an SG-0 does not change the outcome, since it would be like transferring it to a terminal node. The second theorem shows that all SG-1s are equivalent. In addition, it shows that removing the counters simultaneously from two SG-1s does not change the outcome.

These theorems are proved by contradiction and induction using the notion of the simplicity of a position. We say *a position x is simpler than a position y* if there is a sequence of moves that takes y into x.

Lemma 1. *The game with only one counter on non-terminal vertex v is a P-position if, and only if, $g(v) = 1$.*

This follows from the Corollary to Theorem 2 of Ferguson [9] since annihilation plays no role. It can otherwise be easily seen by checking that moving to an SG-1 is a optimal strategy when Condition A is satisfied. It is interesting to note that the only place Condition A is explicitly used in the proofs contained in this paper is in this lemma.

We note the following characterization of the P- and N-positions for misère graph games.

(i) Every terminal position is in N.

(ii) Every follower of a P-position is an N-position.

(iii) Every non-terminal N-position has a P-position as a follower.

THEOREM 1. *If* $x=\{v_1,..., v_n\}$ *and* $y=\{v_1,..., v_n, v_{n+1}\}$, *where* $g(v_{n+1})=0$, *then x and y have the same outcome (i.e., both are N or both are P).*

Proof. Suppose the theorem is false. Let x and y be a simplest counterexample in the sense that no counterexample has a simpler x and for that x no counterexample has a simpler v_{n+1}.

Case 1. $x \in P$ and $y \in N$. Since x is not terminal, neither is y and $y \to y' \in P$.

1a. If the move involves one of the components of x, say $v_n \to v_n'$ without annihilation, then $x' = \{v_1,..., v_n'\} \in N$ and $y' \in P$ gives a simpler counterexample.

1b. If the move annihilates one of the components of x, say $v_n \to v_{n-1}$, then $x' = \{v_1,..., v_{n-2}\} \in N$ and $y' \in P$ is simpler.

1c. If the move annihilates v_{n+1}, say $v_n \to v_{n+1}$, then $y' = \{v_1,..., v_{n-1}\} \in P$ and $x' = \{v_1,..., v_{n-1}, v_{n+1}\} \in N$ is simpler.

1d. If the move involves moving the new component without annihilation, say $v_{n+1} \to v_{n+1}'$, then there exists a further move $v_{n+1}' \to v_{n+1}''$ with $g(v_{n+1}'')=0$. If there is no annihilation, this gives a counterexample with a simpler y. If there is annihilation, say $v_{n+1}''=v_n$, then $y''=\{v_1,..., v_{n-1}\} \in N$ and x is simpler.

1e. If the move involves the new component with annihilation, say $v_{n+1} \to v_n$, then since $g(v_n)$ cannot be zero, there is a move $v_n \to v_n'$ such that $g(v_n')=0$. Whether or not this causes annihilation, the resulting $y' \in P$ and $x' \in N$ is simpler.

Case 2. $x \in N$ and $y \in P$. By Lemma 1, x is not terminal since then y would be in N. Hence there exists a move $x \to x' \in P$.

2a. If the move causes annihilation, say $v_n \to v_{n-1}$, then $x' \in P$ and $y' = \{v_1,..., v_{n-2}, v_{n+1}\} \in N$ is a simpler counterexample.

2b. If the move causes annihilation with v_{n+1}, $v_n \to v_{n+1}$, then $y' = \{v_1,..., v_{n-1}\} \in N$ and $x' = \{v_1,..., v_{n-1}, v_{n+1}\} \in P$ is simpler.

2c. If the move is without annihilation, $v_n \to v_n'$, then $x' \in P$ and $y' = \{v_1,..., v_n', v_{n+1}\} \in N$ is simpler.

In proving the corresponding result about SG-1s, we prove both that any SG-1 may be exchanged for any other, and that two SG-1s may be removed without changing the outcome. These two statements are proved together.

THEOREM 2. *If* $x = \{v_1, \ldots, v_{n-1}, v_n\}$ *and* $y = \{v_1, \ldots, v_{n-1}, v'_n\}$ *with* $g(v_n) = g(v'_n) = 1$, *then x and y have the same outcome. If* $x = \{v_1, \ldots, v_n\}$ *and* $y = \{v_1, \ldots, v_n, v_{n+1}, v_{n+2}\}$ *with* $g(v_{n+1}) = g(v_{n+2}) = 1$, *then x and y have the same outcome.*

Proof. Suppose the theorem is false. Find a counterexample with a simplest x and y with the smallest value of n. There are four cases to consider, whether x is a counterexample to the first or second statement, and whether $x \in P$ and $y \in N$ or vice versa. From Theorem 1, assume without loss of generality that no component of x has SG-value 0.

Case 1. $x = \{v_1, \ldots, v_n\} \in P$ with $g(v_n) = 1$ and $y = \{v_1, \ldots, v_{n-1}, v'_n\} \in N$ with $g(v'_n) = 1$. There exists a move $y \to y' \in P$.

1a. If the move is from a vertex also in x, say $v_{n-1} \to v'_{n-1}$, without annihilation, then $x' = \{v_1, \ldots, v'_{n-1}, v_n\} \in N$ and $y' \in P$ is a simpler counterexample; with annihilation, say $v'_{n-1} = v_{n-2}$, then $x' = \{v_1, \ldots, v_{n-3}, v_n\} \in N$ is simpler; with annihilation of v'_n, say $v'_{n-1} = v'_n$, then $y' = \{v_1, \ldots, v_{n-2}\} \in P$ and $x' = \{v_1, \ldots, v_{n-2}, v'_n, v_n\} \in N$ with $g(v'_n) = g(v_n) = 1$ gives a counterexample with a smaller n.

1b. If the move is from v'_n and causes annihilation, say $v'_n \to v_{n-1}$, then since $g(v_{n-1}) \neq 0$ or 1, there is a move $v_{n-1} \to v'_{n-1}$ such that $g(v'_{n-1}) = 1$. With no annihilation, this gives $x' = \{v_1, \ldots, v_{n-2}, v'_{n-1}, v_n\} \in N$ with $g(v'_{n-1}) = g(v_n) = 1$; with annihilation of v_n, this gives $x' = \{v_1, \ldots, v_{n-2}\} \in N$; with annihilation of say v_{n-2}, this gives $x' = \{v_1, \ldots, v_{n-3}, v_n\} \in N$ with $g(v_{n-2}) = g(v_n) = 1$. This gives a simpler counterexample with $y' = \{v_1, \ldots, v_{n-2}\} \in P$.

1c. If the move is $v'_n \to v''_n$ with no annihilation, $g(v''_n)$ cannot be 0 without contradicting Theorem 1 (since v_n can be moved to an SG-0), so that there is a further move $v''_n \to v'''_n$ with $g(v'''_n) = 1$; with or without annihilation, this gives a simpler counterexample.

Case 2. $x = \{v_1, \ldots, v_n\} \in N$ with $g(v_n) = 1$ and $y = \{v_1, \ldots, v_{n-1}, v'_n\} \in P$ with $g(v'_n) = 1$. This case is symmetrical to Case 1, with x and y interchanged.

Case 3. $x = \{v_1, \ldots, v_n\} \in P$ and $y = \{v_1, \ldots, v_n, v_{n+1}, v_{n+2}\} \in N$ with $g(v_{n+1}) = g(v_{n+2}) = 1$. There exists a move $y \to y' \in P$.

3a. If the move is from a vertex also in x, say $v_n \to v'_n$, then whether or not there is annihilation, $x' = \{v_1, \ldots, v'_n\} \in N$ and $y' \in P$ is simpler.

3b. If the move is from one of the additional vertices of y with annihilation, say $v_{n+1} \to v_n$, then since $g(v_n) \neq 0$ or 1, there is a move $v_n \to v_n'$ with $g(v_n') = 1$ taking x into $x' = \{v_1, \ldots, v_n'\} \in N$. With or without annihilation this is simpler.

3c. If the move is $v_{n+1} \to v_{n+1}'$ without annihilation, then since $g(v_{n+1}') \neq 0$ or 1, there is a move $v_{n+1}' \to v_{n+1}''$ with $g(v_{n+1}'') = 1$. With or without annihilation this gives a simpler $y'' \in N$.

Case 4. $x = \{v_1, \ldots, v_n\} \in N$ and $y = \{v_1, \ldots, v_n, \; v_{n+1}, \; v_{n+2}\} \in P$ with $g(v_{n+1}) = g(v_{n+2}) = 1$. x cannot be terminal since y can be moved to $y' = \{v_1, \ldots, v_n, v_{n+1}, v_{n+2}'\} \in N$ with $g(v_{n+2}') = 0$, which contradicts Lemma 1, Theorem 1. Hence, there exists a move $x \to x' \in P$. The same move in y, say $y \to y' \in N$, gives a simpler counterexample whether or not there is annihilation.

As a corollary to Theorems 1 and 2, we have a characterization of the P-positions among these positions with counters only on vertices having SG-values 0 or 1.

COROLLARY. *Under Condition* A, *if* $x = \{v_1, \ldots, v_n\}$ *with each* $g(v_i) = 0$ *or* 1, *then* x *is a P-position if and only if* $g(v_i) = 1$ *for an odd number of* v_i.

Of course, the optimal strategy and hence simple direct proof of this corollary is clear. If your opponent moves from a position of this form, you can always return it to a position of this form. Your opponent will never be left without a move.

Theorem 1 implies that there is only one type of vertex with SG-value 0. We may as well assume that every vertex of SG-value 0 is terminal; that is, the outcome of the game is not changed if all edges leading out of vertices with SG-value 0 are deleted. Similarly by Theorem 2, there is only one type of vertex of SG-value 1. The outcome of the game is not changed if all edges leading out of each SG-1 are deleted except for one edge, that edge going to a terminal vertex.

Together, these theorems imply that the whole structure of the game on the graph is determined by the connections between the other vertices, that is, those vertices of SG-value 2 or greater. We call the subgraph on the set V_1 of vertices of SG-value 2 or greater the *reduced graph*.

3. Two Characterization Theorems

We shall now prove two general theorems characterizing the set, P, of P-positions in the game under conditions involving only the connections between vertices with SG-values 2 or greater. The proofs use the standard procedure of verifying statements (i), (ii), and (iii) of the previous section.

In Theorem 3, we find the P-positions provided the graph satisfies the following additional condition. For each even integer k, a vertex v of SG-value k is called a *terminal* k if it does not have a follower of SG-value $k+1$. For each even $k \geqslant 2$, and for each terminal k, v, form the set W by starting with $W = \{v\}$ and then recursively joining to W all vertices of SG-value k or $k+1$ that have a follower in W. Label these sets $W_1, W_2, \ldots,$ in some order. Every SG-j for $j \geqslant 2$ belongs to at least one of the sets W_i.

CONDITION B. *The sets $W_1, W_2, \ldots,$ are disjoint, and for every vertex v of SG-value $2n$ or $2n+1$ for $n > 1$ and for every k, $1 \leqslant k < n$, either* B1: *there exist some W_i and $w_1 \in W_i$ and $w_2 \in W_i$ with $v \to w_1$,*

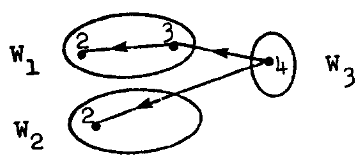
$$v \to w_2, g(w_1) = 2k \qquad \textit{and} \qquad g(w_2) = 2k+1,$$

or B2: for $m = 2k$ and $m = 2k+1$, there exist w_1 and w_2 in distinct

$$W_i \textit{ with } v \to w_1, v \to w_2 \qquad \textit{and} \qquad g(w_1) = g(w_2) = m.$$

For Nim, with vertices $\{0, 1, 2, \ldots\}$ and SG-function $g(v) = v$, the sets $W_1, W_2, \ldots,$ are $\{2, 3\}$, $\{4, 5\}$, $\{6, 7\}, \ldots,$ and so satisfy Condition B.

The main part of Condition B is that the sets W_i be disjoint. If the W_i are disjoint, the rest of Condition B is also necessary for Theorem 3 in the sense that on such graphs positions x can be found for which the conclusion of Theorem 3 is not true. As a counterexample, consider the reduced graph



The position x consisting of the vertex of SG-value 4 and the lower vertex of SG-value 2 is a P-position with SG-value 6.

THEOREM 3. *Under Condition* B, $P = Q_1 \cup Q_0$ *where*

$Q_1 = \{x \colon g(x) = 1$ and each set W_i contains an even number of components of $x\}$,

$Q_0 = \{x \colon g(x) = 0$ and some set W_i contains an odd number of components of $x\}$.

Proof. (i) Clearly no terminal x is in P.

(ii) Suppose $x \in Q_1$, $x \to y$, and $g(y) = 0$. A change $0 \to 1$ or $1 \to 0$ in the SG-value occurs if and only if one of the components is moved from SG-value $2j$ to $2j+1$, or from SG-value $2j+1$ to $2j$ for some $j \geqslant 0$ with or without annihilation. Since the sets W_i are disjoint, y also has an even number of components in each W_i, and so is not in Q_0. On the other hand, if $x \in Q_0$, $x \to y$ and $g(y) = 1$, the same argument works to show that y must have an odd number of components in some W_i and so is not in Q_1.

(iii) Suppose $x \notin Q_1 \cup Q_0$, and x is not terminal.

Case 1. $g(x) \geqslant 2$. There exists a move $x \to y$ with $g(y) = 0$ involving the moving of some component of x from one vertex to a follower v' of lesser SG-value. If $y \in Q_0$, we are done. If not, y has an even number of components in each W_i. Therefore, move the indicated component either to a vertex of the same SG-value as v' in another W_j (case B_2) or to a vertex of SG-value one greater or one less than v' depending on whether $g(v')$ is even or odd (in the same W_j (case B1) if its SG-value is greater than one).

Case 2. $g(x) = 1$ *and some* W_i *contains an odd number of components of* x. Since $g(x)$ is odd, there is at least one component with odd SG-value. Moving any such component to a vertex with SG-value 1 less does not disturb the parity in the W_i with or without annihilation and puts x into Q_0.

Case 3. $g(x) = 0$ *and there is an even number of components of* x *in each* W_i *and* x *is not terminal.* If there exists a component of x with odd SG-value, then any move from SG-value $2j+1$ to $2j$ $(j \geqslant 0)$ changes the SG-value of x by 1 without disturbing the parity in the W_i, and puts x into Q_1. If all components of x have SG-value 0 then $x \in N$ by the Corollary of Theorem 2. Otherwise, all components of x have even SG-values, at least one of SG-value 2 or greater, so for some $j \geqslant 1$ there are at least two components of x with SG-values $2j$ in the same W_i. At least one of them has a follower of SG-value $2j+1$. This move would put x into Q_1.

We now assume that the graph has no SG-values greater than 3, and obtain in Theorem 4 a considerable improvement over the previous theorem.

Let V_1 denote the set of vertices of SG-value greater than 1. Every graph that has an SG-2 has a terminal 2. A terminal 2 is a terminal vertex of the reduced graph. For a given terminal 2, say v, we may construct a set, W, generated by v by first putting $W = \{v\}$ and then adding vertices to W by the following rule until no more vertices are added.

RULE. *Add to* W *all vertices* $w \in V_1$ *such that*

(1) *w has at least one follower in W, and*

(2) *if $w' \in V_1$ is a follower of w not in W, then there is a follower of w' that is in W.*

Each set so constructed contains exactly one terminal 2. Let us denote the sets so constructed by $W_1, W_2, \ldots,$ finite or countably infinite, one for each terminal 2. Let R denote the set of remaining vertices in V_1

$$R = V_1 - \bigcup W_i.$$

It is important to note that the sets W_i so constructed are disjoint. In fact, there is not even any communication between the sets W_i, that is, there is no edge joining a vertex of W_i to W_j if $i \neq j$. If there were, there would be a simplest $v \in W_i$ such that $v \to w \in W_j$; but by the Rule for constructing W_i, there would be a $v' \in W_i$ such that $w \to v'$, and by the Rule for constructing W_j, there would be $w' \in W_j$ such that $v' \to w'$; thus, v would not be the simplest.

For arbitrary disjoint sets $U_1, U_2, \ldots,$ of vertices define

$$E(U_1, U_2, \ldots,) = \{x\colon x \text{ has an even number of components in each } U_i$$
$$\text{and none in } V_1 - \bigcup U_i\}.$$

Note that $E(U_1) \subset E(U_1, U_2)$.

The following theorem states under certain conditions that the set of P-positions has the form $P = Q_1 \cup Q_0$ where

$$Q_1 = \{x\colon g(x) = 1 \text{ and } x \in E(W_1, W_2, \ldots,)\}$$
$$Q_0 = \{x\colon g(x) = 0, x \text{ non-terminal and } x \not\to Q_1\}$$

where $x \not\to Q_1$ stands for the statement "x does not have a follower in Q_1." The following lemma gives a better understanding of the set Q_0.

Lemma 2. *Let $Q_1 = \{x\colon g(x) = 1$ and $x \in E(W_1, W_2, \ldots,)\}$. Let x be non-terminal with $g(x) = 0$. Then $x \to Q_1$, if and only if (1) $x \in E(W_0, W_1, W_2, \ldots,)$, where W_0 consists of some annihilable pair in R, or (2) for some $j \geq 1$ and $v \in R$ with a follower in W_j, $x \in E(W_1, \ldots, W_j + \{v\}, \ldots,\}$.*

Proof. *If.* If x satisfies (1), annihilate the pair if there is one in x. Else any SG-1 $\to$ 0 or SG-0 $\to$ 1 works. Else there exists two elements in some W_i. At most one is a terminal 2. The other must have a follower in W_i. If x satisfies (2) but not (1), then $v \in x$, so move $v \to W_j$.

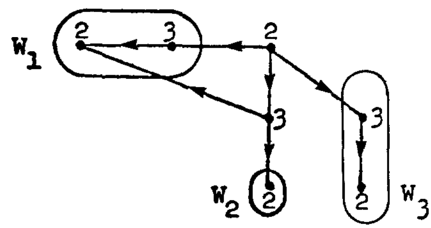
Only if. If the move $x \to Q_1$ involves moving an SG-1 $\to$ 0 or SG-0 $\to$ 1, then $x \in E(W_1, W_2, \ldots,)$. Otherwise, the move involves moving an SG-3 $\to$ 2 or an SG-2 $\to$ 3. x cannot have 3 or more components in R. If it has 2, they must be an annihilable pair as in (1). If it has 1, then either it is movable into some W_j as in (2), or some move from some W_j can annihilate it, in which case it is movable into that W_j by part (2) of the construction of W_j, so x is as in (2). If it has none, then x must be in $E(W_1, W_2, \ldots,)$ since there is no communication among the W_i.

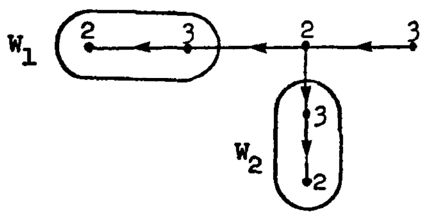
CONDITION C. C1. *If (u, v) are an annihilable pair in R, then they do not both have followers in the same set W_i.*

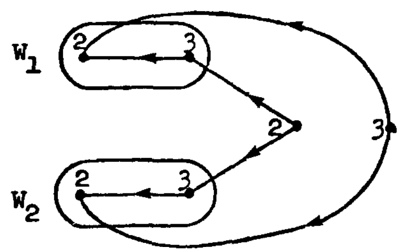
C2. *If v is an SG-3 in R with followers only in R, then there is no W_j such that all followers of v have a follower in W_j.*

C3. *There does not exist an SG-2 in R and an SG-3 in R with followers only in the same identical set of W_i's, and no followers in R.*

This condition is much weaker than Condition B restricted to graphs with SG-values no greater than 3, but it is still quite restrictive. Here are the reduced graphs of three simple examples that violate Conditions C1, C2 and C3, respectively,



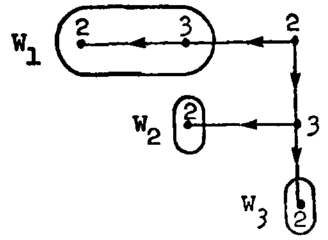




The applications of Theorem 4 made in the next section are all of a special type in which the graph satisfies the simpler condition:

CONDITION C′. *No SG-2 in R has a follower in $\bigcup W_i$.*

It is easy to see that Condition C′ implies Condition C. Here is the reduced graph of a counterexample to the converse:



THEOREM 4. *Suppose the graph has no vertex of SG-value greater than* 3 *and assume Condition* C *holds. Then,* $P = Q_1 \cup Q_0$, *where*

$$Q_1 = \{x\colon g(x) = 1 \text{ and } x \in E(W_1, W_2, \ldots,)\}$$

$$Q_0 = \{x\colon g(x) = 0 \text{ and } x \text{ non-terminal and } x \not\to Q_1\}.$$

Proof. (i) There are no terminal positions in P.

(ii) Let $x \in Q_1$, $x \to y$ and $g(y) = 0$. Since the SG-value has changed by 1, the parity of x in the sets W_i does not change (since they do not communicate), unless some w in some W_j moved to w' in R, in which case the Rule implies that there is a w'' in W_j, so that $w' \to w''$ puts y back into Q_1; so $y \notin Q_0$.

Let $x \in Q_0$, $x \to y$. Then $y \notin Q_1$ by the definition of Q_0,

(iii) Suppose $x \notin Q_1 \cup Q_0$.

Case 1. $g(x) \geqslant 2$.

1a. *x has* 3 *or more components in* R. Removing one of them (i.e., putting it to SG-0 or 1) so that the remaining two (or more) are not an annihilable pair in R, puts x into Q_0 by Lemma 2.

1b. *x has* 2 *components in* R. If not an annihilable pair, removing one of the components of some W_i (which exists because $g(x) \geqslant 2$ implies there is an odd number of components in V_1) can put x into Q_0. If an annihilable pair, one of them can be removed to put the position into Q_0 by Condition C1 and Lemma 2.

1c. *x has* 1 *component in* R. Removing this component to SG-1 or 0 depending on whether or not the result is in $E(W_1, W_2, \ldots,)$ puts x into $Q_1 \cup Q_0$.

1d. *x has no components in* R. Moving any SG-2 or 3 to SG-0 or 1 depending on whether the result is in $E(W_1, W_2, \ldots,)$ or not puts x into $Q_1 \cup Q_0$.

Case 2. $g(x) = 1$ *and* $x \notin E(W_1, W_2, \ldots,)$. We must show we can move $x \to y$ with $g(y) = 0$ where y is not in one of the two categories of Lemma 2.

2a. *x has no components in R.* Moving any SG-1 $\to$ 0 or any SG-3 $\to$ 2 within some W_i puts x into Q_0.

2b. *x has 1 component in R.* If x satisfies (2) of Lemma 2 with component $v \in R$, by the Rule, there exists a move $v \to v' \in V_1 - W_j$ such that $v' \not\to W_j$. Since the move $v \to v'$ changes an SG-2 to SG-3 or conversely, the result will be in Q_0. Otherwise, move any SG-1 $\to$ 0 or any SG-3 $\to$ 2 within some W_i. If such a move does not exist, the element of R is an SG-3; moving it to the correct SG-2 using Condition C2 puts x into Q_0.

2c. *x has 2 components in R.* If not an annihilable pair, moving any SG-1 $\to$ 0 or any SG-3 $\to$ 2 within some W_i puts x into Q_0. If such a move does not exist, there is one SG-3 and one SG-2 in R. If the SG-3 or the SG-2 can be moved within R, such a move puts x into Q_0. Otherwise Condition C3 implies we can move one of them into some W_j to put x into Q_0. If the two components of R are annihilable, Condition C1 implies we can move one of them into some W_j to put x into Q_0.

2d. *x has 3 or more components in R.* Moving any SG-1 $\to$ 0 or any SG-3 $\to$ 2 within some W_i puts x into Q_0. If such a move does not exist, x has an odd number of SG-3*s* in R. If x has 3 or more SG-3*s*, moving any one of them to an SG-2 puts x in Q_0. Otherwise, x has a unique SG-3 (in R). If this SG-3 has a follower in some W_i or an unoccupied follower in R, moving it there puts x in Q_0. Else an occupied follower in R is an SG-2 which may be moved to R or to at least two different W_i, one of which will put x in Q_0.

Case 3. *$g(x) = 0$ and x is non-terminal and $x \notin Q_0$.* Then x can be moved to Q_1 by definition of Q_0.

Two special cases of this theorem are of interest. The first case occurs when R is empty. Then Theorem 4 becomes a special case of Theorem 3, and the simpler description of Q_0 given there is valid.

The second case occurs when each set W_i consists of one element, namely the singleton 2. Then, since an even number of elements in a singleton set means no elements, the description of Q_1 and Q_0 simplifies to

$$Q_1 = \{x: g(x) = 1 \text{ and } x \text{ has no } SG\text{-}2s \text{ or } SG\text{-}3s\}$$

$$Q_0 = \{x: g(x) = 0 \text{ and } x \text{ has at least one } SG\text{-}2 \text{ or } SG\text{-}3 \text{ but not only an annihilable pair}\}.$$

4. Applications to Subtraction Games

Let S, be a non-empty subset of the positive integers. *A subtraction game with subtraction set S*, denoted by G_S, is the game played on the graph whose vertices are the non-negative integers and whose directed edges include an edge from k to j if and only if $k-j \in S$. Nim is the subtraction game G_S when S is the set of all positive integers.

That all subtraction games satisfy Condition A is proved in Ferguson [9]. To see how successful Theorems 3 and 4 are in solving subtraction games, we will investigate the subtraction sets $S_k = \{1, 2, \ldots, k\}$, and all subtraction games G_S with $S \subset \{1, 2, \ldots, 7\}$, plus a few more with SG-sequences identical to one of these. A table of these, due to Austin [13], may be found in Berlekamp, Conway and Guy [14], which is essentially duplicated in Table I.

All subtraction games with finite S have eventually periodic SG-sequences. For a given S, it may happen for some $k \notin S$ that $g(n) \neq g(n+k)$ for all $n \geqslant 0$, in which case k may be added to S without changing the SG-sequence. In particular, provided the period starts at 0 (which occurs for all S in Table I except $S = \{2, 4, 7\}$), if $s \in S$ then $s + \lambda$ can be added where λ is the period, and if k can be added so can $k + \lambda$. For each S in the table, all other additions that may be made to S without changing the SG-sequence are indicated. It is noted later that the SG-sequence alone does not determine the solution.

4.1. *The Subtraction Sets* $S_k = \{1, 2, \ldots, k\}$

For the subtraction set S_k, the SG-sequence is

$$\begin{array}{rcccccccccc} v: & 0 & 1 & 2 \cdots k & k+1 & k+2 \cdots & 2k+1 & 2k+2 \cdots \\ g(v): & 0 & 1 & 2 \cdots k & 0 & 1 \quad \cdots & k & 0 \quad \cdots \end{array}$$

cyclic of period $\lambda = k+1$. It is easy to see that Condition B is satisfied since every vertex $v \geqslant k$ has a follower of every SG-value from 0 to k excluding its own SG-value.

For odd k, the sets W_i may be taken to consist of all vertices of SG-value $2i$ or $2i+1$ for $i = 1, 2, \ldots, (k-1)/2$. Thus for odd k, the P-positions are $P = Q_1 \cup Q_0$, where

$$Q_1 = \{x: g(x) = 1 \text{ and } x \in E(W_1, \ldots, W_{(k-1)/2})\}$$
$$Q_0 = \{x: g(x) = 0 \text{ and } x \notin E(W_1, \ldots, W_{(k-1)/2})\}.$$

In particular, for S_3 and S_5 the description of Q_1 and Q_0 simplifies to

$$Q_1 = \{x: g(x) = 1\}$$
$$Q_0 = \varnothing.$$

This is because for S_3 and S_5, $g(x) = 1$ implies that there are an even number of SG-2s and 3s and an even number of SG-4s and 5s in x. For S_7, however, this simplification does not take place because $x = \{2, 4, 6\} \in Q_0$.

TABLE I
Subtraction Games with Subtraction Sets Contained in {1, 2, 3, 4, 5, 6, 7}

S	SG-Sequence	Period λ	Additions[a]	
{1}	0101	2		S_1
{2}	00110011	4		S_1
{1, 2}	012012	3		S_2
{3}	000111000111	6		S_1
{2, 3}	00112	5		S_2
{1, 2, 3}	0123	4		S_3
{4}	00001111	8		S_1
{1, 4}	01012	5		S_2
{2, 4}	001122	6	3	S_2
{3, 4}	0001112	7		S_2
{1, 3, 4}	0101232 .	7	6	§4.2
{1, 2, 3, 4}	01234	5		S_4
{5}	0000011111	10		S_1
{2, 5}	001102	7		S_2
{3, 5}	00011122	8	4	S_2
{4, 5}	000011112	9		S_2
{1, 4, 5}{1, 3, 4, 7}	01012323	8	3, 5, 7	S_3
{2, 3, 5}{2, 4, 5}	0011223	7	3, 4	§4.3
{1, 2, 3, 4, 5}	012345	6		S_5
{6}	000000111111	12		S_1
{1, 6}	0101012	7		S_2
{3, 6}	000111222	9	4, 5	S_2
{4, 6}	0000111122	10	5	S_2
{5, 6}	00000111112	11		S_2
{1, 2, 6}	0120123	7	5	§4.3
{1, 3, 6}	010101232	9	8	§4.2
{1, 5, 6}	01010123232	11	3, 8, 10	§4.2
{2, 3, 6}	001120312	8		§4.2
{2, 4, 6}{2, 3, 5, 6}	00112233	8	3, 4, 5	§4.4
{2, 5, 6}	00110213021	11	9	§4.2
{1, 2, 4, 6}{1, 2, 6, 7}	01201234	8	4, 7	§4.10
{1, 4, 5, 6}	010123234	9	3, 8	S_4
{1, 2, 4, 5, 6}	0120123453	10	8, 9	§4.10
{1, 2, 3, 4, 5, 6}	0123456	7		S_6
{7}	00000001111111	14		S_1
{2, 7}	001100112	9		S_2
{3, 7}	0001110221	10		S_2
{4, 7}	00001111222	11	5, 6	S_2
{5, 7}	000001111122	12	6	S_2
{6, 7}	0000001111112	13		S_2
{1, 4, 7}	01012012	8		S_2
{1, 6, 7}	010101232323	12	3, 5, 9	S_3
{2, 4, 7}	00112203102	3	10[a]	§4.5
{2, 5, 7}	0011021322031001122332	22	11, 15, 17, 20	§4.6
{2, 6, 7}	0011001120312	13	11	§4.2
{3, 4, 7}{3, 5, 7}{3, 6, 7}	0001112223	10	4, 5, 6	§4.3
{1, 4, 6, 7}	0101201232012	13	9, 12	§4.7
{2, 3, 4, 7}	00112203142	11	8, 9	§4.10
{2, 3, 5, 7}{2, 4, 5, 7}{2, 4, 6, 7}	001122334	9	3, 4, 5, 6	§4.10
{2, 5, 6, 7}	001102132233	12	10	§4.8
{1, 2, 5, 6, 7}	01201234534	11	4, 9, 10	§4.10
{1, 3, 4, 6, 7}{1, 4, 5, 6, 7}	0101232345	10	3, 5, 9	§4.9
{1, 2, 3, 4, 5, 6, 7}	01234567	8		S_7

[a] Additions to S that do not change the SG-sequence include $s + n\lambda$ with $n \geqslant 0$ for $s \in S$ or s an element of this column. Exception: For $S = \{2, 4, 7\}$, the only additions are 10, 13, 16,....

For even k, the sets W_i are as above for $i = 1, 2, \ldots, (k-2)/2$, but each vertex of SG-value k has no follower of SG-value $k+1$ so that each SG-k forms a set W_i by itself $i = k/2, k/2+1, \ldots$. Hence for even k, $P = Q_1 \cup Q_0$, where

$$Q_1 = \{x\colon g(x) = 1 \text{ and } x \in E(W_1, \ldots, W_{(k-2)/2})\}$$

$$Q_0 = \{x\colon g(x) = 0 \text{ and } x \notin E(W_1, \ldots, W_{(k-2)/2})\}.$$

In particular, S_2, S_4 and S_6 have simplified statements. For S_k and $k = 2$, 4, or 6,

$$Q_1 = \{x\colon g(x) = 1 \text{ and } x \text{ has no component of SG-value } k\},$$

$$Q_0 = \{x\colon g(x) = 0 \text{ and } x \text{ has at least one component of SG-value } k\}.$$

The solution for S_2 works for any graph game whose graph satisfies Condition A and has only SG-values 0, 1, and 2. From Table I, these include games with subtraction sets consisting of one or two elements, and the game with $S = \{1, 4, 7\}$, plus supersets with the same SG-sequence, such as $\{1, 3, 5\}$, $\{1, 2, 4\}$, $\{1, 2, 7, 8\}$, etc. This solution is essentially the same as the solution of the corresponding non-annihilation misère subtraction games.

The solution for S_3 works for any graph game whose graph has vertices of SG-value no greater than 3 provided Condition A is satisfied and provided there is only one terminal 2. These include $\{1, 2, 3\}$, $\{1, 3, 4, 6\}$, $\{1, 4, 5\}$, $\{1, 3, 6, 8\}$, $\{1, 5, 6, 8\}$, $\{2, 3, 6, 7\}$, $\{2, 5, 6, 9\}$, $\{1, 6, 7\}$, $\{2, 6, 7, 11\}$, $\{1, 3, 4, 7\}$ and the supersets with the same SG-sequence.

The solution for S_4 works for any graph game whose graph has vertices of SG-value no greater than 4 provided Condition A is satisfied and provided there is only one terminal 2. These include $\{1, 2, 3, 4\}$, $\{1, 4, 5, 6\}$ and supersets with the same SG-sequence.

The solution for S_5 works for any graph game whose graph has vertices of SG-value no greater than 5 provided Condition A is satisfied, there is only one terminal 2, and there is only one terminal 4. These include $\{1, 2, 3, 4, 5\}$, $\{1, 3, 4, 6, 7, 9\}$, $\{1, 4, 5, 6, 7, 9\}$ and supersets with the same SG-sequence.

4.2. $\{1, 3, 4\}$ *and Supersets with the Same SG-Sequence*

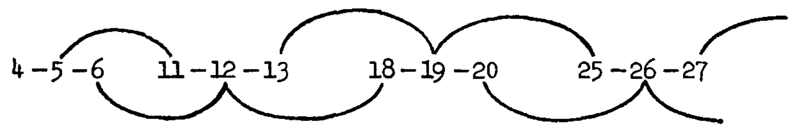
That the SG-sequence alone does not determine the solution can be seen by considering the subtraction set $\{1, 3, 4\}$ and various supersets with the same SG-sequence. Results of this analysis apply equally well to the subtraction sets $\{1, 3, 6\}$, $\{1, 5, 6\}$, $\{2, 3, 6\}$, $\{2, 5, 6\}$, and $\{2, 6, 7\}$ and their supersets with obvious modifications.

EXAMPLE 1. $\{1, 3, 4\}$. The SG-sequence of $\{1, 3, 4\}$ is contained in Table I. The reduced graph is

$$4 — 5 — 6 \quad 11 — 12 — 13 \quad 18 — 19 — 20 \cdots$$

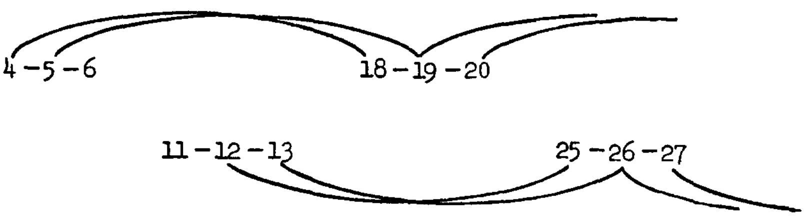
where the direction of the edges is always from the larger number to the smaller. The terminal 2's are vertices 4, 11, 18,..., $7k-3$,.... Each of these generates a set W_i consisting of three elements. $W_1 = \{4, 5, 6\}$, $W_2 = \{11, 12, 13\}$,..., $W_k = \{7k-3,\ 7k-2,\ 7k-1\}$,.... The set R is empty, thus leading to a set of P-positions described in Theorem 3.

EXAMPLE 2. $\{1, 3, 4, 6\}$. The reduced graph becomes



This yields a unique terminal 2 and a set W_1 identical to V_1. The solution is the same as that for S_3.

EXAMPLE 3. $\{1, 3, 4, 13\}$. The reduced graph is

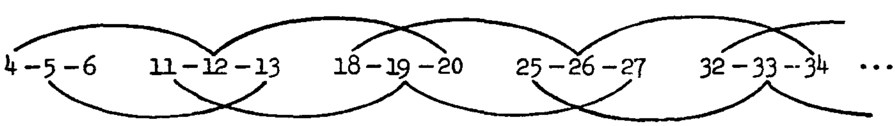


There are two terminal 2s leading to $W_1 = \{4, 5, 6, 18, 19, 20,...,\}$ and $W_2 = \{11, 12, 13, 25, 26, 27,...,\}$, and an empty R. The set of P-positions is given in Theorem 3.

EXAMPLE 4. $\{1, 3, 4, 20\}$. The additional move now breaks V_1 up into three pieces, $W_1 = \{4, 5, 6, 25, 26, 27,...,\}$, $W_2 = \{11, 12, 13,...,\}$ and $W_3 = \{18, 19, 20,...\}$, and again R is empty. Similarly, one can handle sets of the form $\{1, 3, 4, 7k-1\}$.

EXAMPLE 5. $\{1, 3, 4, 13, 20\}$. Now there are two terminal 2s, namely, 4 and 11, but the sets they generate, namely, $W_1 = \{4, 5, 6, 18, 19, 20, 39\}$ and $W_2 = \{11, 12, 13\}$, do not satisfy Condition C. Condition C1 is violated by the annihilable pairs $\{25, 26\}$ and $\{32, 33\}$. See the reduced graph in Section 4.11 where the solution is given.

EXAMPLE 6. $\{1, 3, 4, 8\}$, The reduced graph is



The terminal 2s are 4, 11, 18, 25, 32,.... In forming the sets W_i only the first gets larger. The sets are $W_1 = \{4, 5, 6, 13\}$, $W_2 = \{11\}$, $W_3 = \{18\}$,.... We may lump the sets W_2, W_3,..., into R to simplify the description of the sets Q_1 and Q_0, where $R = \{12, 19, 20, 26, 27,...,\}$. Let

$$R' = R \cup W_2 \cup W_3 \cdots = \{11, 12, 18, 19, 20, 25, 26, 27,...\}.$$

Then

$$Q_1 = \{x: g(x) = 1 \text{ and } x \text{ has no components in } R'\}$$
$$Q_0 = \{x: g(x) = 0 \text{ and } x \text{ has at least one component in } R' \text{ other than just the vertex 12 or just a single annihilable pair in } R'\}.$$

4.3. $\{1, 2, 6\}$

The reduced graph is

$$2 \quad 5-6 \quad 9 \quad 12-13 \quad 16 \quad 19-20 \quad .$$

Thus $W_1 = \{5, 6, 12, 13, 19, 20,...,\}$, $W_2 = \{2\}$, $W_3 = \{9\}$, $W_4 = \{16\}$,..., $R = \emptyset$. Let $R' = \bigcup_2^\infty W_i = \{2, 9, 16,...\}$. With this notation, Q_1 and Q_0 become

$$Q_1 = \{x: g(x) = 1 \text{ and } x \text{ has no components in } R'\},$$

$$Q_0 = \{x: g(x) = 0 \text{ and } x \text{ has at least one component in } R'\}.$$

Any supeset of $\{1, 2, 6\}$ with the same SG-sequence has the same reduced graph and hence the same solution. Similarly for the solutions of $\{2, 3, 5\}$, $\{2, 4, 5\}$, $\{3, 4, 7\}$, $\{3, 5, 7\}$, $\{3, 6, 7\}$ and supersets with the same SG-sequence.

4.4. $\{2, 4, 6\}$ *and Supersets with the Same SG-Sequence.*

A similar analysis may be made of $\{3, 6, 9\}$, $\{2, 4, 6, 8\}$, etc.

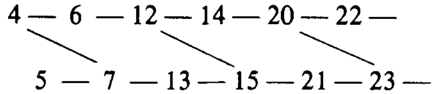
EXAMPLE 1. $\{2, 4, 6\}$. The reduced graph is

$$4 — 6 — 12 — 14 — 20 — 22 —$$
$$5 — 7 — 13 — 15 — 21 — 23 —$$

The terminal 2s generate $W_1 = \{4, 6, 12, 14,...,\}$ and $W_2 = \{5, 7, 13, 15,...,\}$ and R is empty. The P-positions are

$$Q_1 = \{x: g(x) = 1 \text{ and } x \in E(W_1, W_2)\},$$
$$Q_0 = \{x: g(x) = 0 \text{ and } x \notin E(W_1, W_2)\}.$$

EXAMPLE 2. $\{2, 3, 4, 6\}$. The reduced graph is

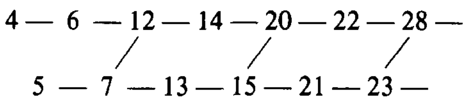


The terminal 2s generate $W_1 = \{4, 6, 12, 14, \ldots,\}$ and $W_2 = \{5\}$, with $R = \{7, 13, 15, \ldots,\}$. With $R' = W_2 \cup R$, the P-positions are

$Q_1 = \{x: g(x) = 1$ and x has no components in $R'\}$.

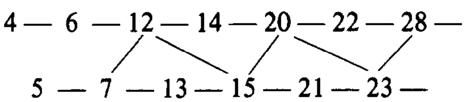
$Q_0 = \{x: g(x) = 0$ and x has at least one element in R' other than a single element of $\{7, 15, 23, \ldots,\}$ and other than a single annihilable pair of $R'\}$.

EXAMPLE 3. $\{2, 4, 5, 6\}$. The reduced graph is



The terminal 2s generate $W_1 = \{4, 6\}$ and $W_2 = \{5, 7, 13, 15, \ldots,\}$ with $R = \{12, 14, 20, 22, \ldots,\}$. This graph does not satisfy Condition C2 at 14. The solution to this game is presented in Section 4.11.

EXAMPLE 4. $\{2, 3, 4, 5, 6\}$ or $\{2, 3, 5, 6\}$. The reduced graph is



The sets W_i and R are the same as for Example 2, and the P-positions for the game are also the same.

4.5. $\{2, 4, 7\}$

This is the only SG-sequence whose period does not start at 0. The reduced graph on the set $V_1 = \{4, 5, 7, 10, 13, \ldots,\}$ has a single edge joining 7 to 5. We may take $R' = \{4, 10, 13, \ldots,\}$ and then the sets Q_1 and Q_0 are as stated for $\{1, 2, 6\}$.

4.6. $\{2, 5, 7\}$ *and Supersets with the Same SG-Sequence*

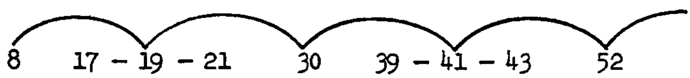
EXAMPLE 1. $\{2, 5, 7\}$. The reduced graph is

5 — 7 — 9 — 11 — 18 — 20 — 27 — 29 — 31 — 33 — 40 — 42 —

8 17 — 19 — 21 30 39 — 41 — 43

The sets W_i are $W_1 = \{5, 7, 9, 11,...,\}$, $W_2 = \{8\}$, $W_3 = \{17, 19, 21\},...,$ and R is empty, leading to the simplified description of Q_1 and Q_0 found in Theorem 3. Supersets of $\{2, 5, 7\}$ with the same SG-sequence do not affect the set W_1 materially, but may change the other W_i.

EXAMPLE 2. $\{2, 5, 7, 11\}$. Except for W_1 the reduced graph is

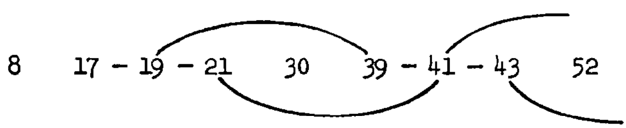


The terminal 2s are 8, 17, 39,..., each forming its own set W_i. Therefore we may let $R' = \{8, 17, 19, 21, 30,...,\}$ and the P-positions are

$Q_1 = \{x: g(x) = 1$ and x has no components in $R'\}$,

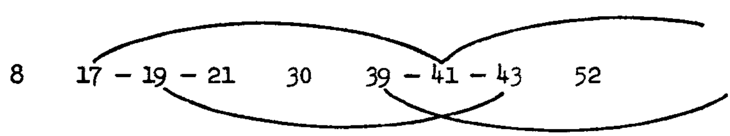
$Q_0 = \{x: g(x) = 0$ and x has at least one component in R' other than a single annihilable pair of $R'\}$.

EXAMPLE 3. $\{2, 5, 7, 20\}$. Except for W_1 the reduced graph is



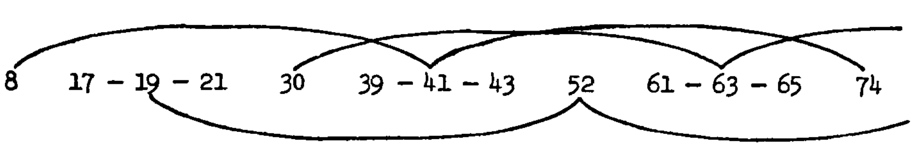
There is one new big $W_2 = \{17, 19, 21, 39, 41, 43,...,\}$ treated like $\{1, 3, 4, 6\}$.

EXAMPLE 4. $\{2, 5, 7, 24\}$. Except for W_1 the reduced graph is

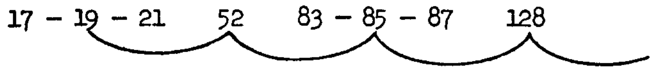


the new joined piece is now treated like $\{1, 3, 4, 8\}$.

EXAMPLE 5. $\{2, 5, 7, 33\}$. Except for W_1 the reduced graph is



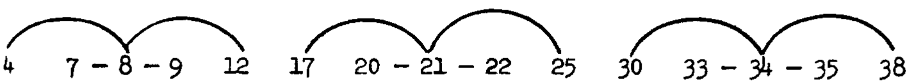
There are three pieces. Two of them, beginning with 8 and with 30 are treated like $\{2, 5, 7, 11\}$. The third is



giving rise to a $W_2 = \{17, 19, 21, 52\}$. This and W_1 are the only W_i to consist of more than one element.

4.7. $\{1, 4, 6, 7\}$ *and Supersets with the Same SG-sequence*

The reduced graph for the set $\{1, 4, 6, 7\}$ is



there are many terminal 2s, namely, 4, 7, 17, 20, 30, 33,..., but each forms its own set W_i. Therefore, we may take $R' = R \cup UW_i = V_1$, and P-positions may be described as

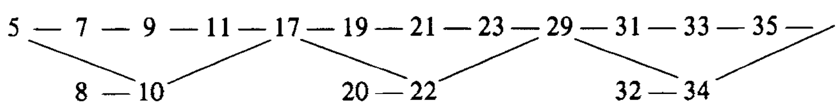
$$Q_1 = \{x: g(x) = 1 \text{ and all components are SG-0s or 1s}\},$$

$$Q_0 = \{x: g(x) = 0 \text{ and at least one SG-2 or SG-3 other than a single annihilable pair}\}.$$

The various supersets of $\{1, 4, 6, 7\}$ with the same SG-sequence have quite different reduced graphs, but it is not difficult to see they all have the same P-positions, with the understanding that the definition of an annihilable pair may change. The reason is that it is impossible to create an SG-3 whose only follower in the reduced graph is a terminal 2. Therefore, each W_i must consist of a single element.

4.8. $\{2, 5, 6, 7\}$ *and Supersets with the Same SG-Sequence*

The reduced graph for the set $\{2, 5, 6, 7\}$ is



The terminal 2s generate $W_1 = \{5, 7, 9, 11, 17, 19,...,\}$, $W_2 = \{8\}$, $W_3 = \{20\},...,$ with $R = \{10, 22, 34,...,\}$. If we define $R' = R \cup W_2 \cup W_3 \cdots$, then

$$Q_1 = \{x: g(x) = 1 \text{ and } x \text{ has no components in } R'\},$$

$$Q_0 = \{x: g(x) = 0 \text{ and } x \text{ has at least one component in } R' \text{ other than a single element of } \{10, 22, 34,...,\} \text{ or a single annihilable pair in } R'\}.$$

The various supersets of $\{2, 5, 6, 7\}$ with the same SG-sequence may have different graphs with a different set of terminal 2s and a different R. However, the description given above for Q_1 and Q_0 still holds with the understanding that the definition of an annihilable pair may change. For example, the reduced graph for $\{2, 5, 6, 7, 10\}$ differs from the above significantly only in that there are now edges from 20 to 10, from 32 to 22, etc., thus giving more possible annihilable pairs of R'.

4.9. $\{1, 3, 4, 6, 7\}$ (*and Similarly* $\{1, 4, 5, 6, 7\}$)

As an application of Theorem 3 to a game other than that given by S_k, consider the subtraction set $\{1, 3, 4, 6, 7\}$. The SG-sequence is

v:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$g(v)$:	0	1	0	1	2	3	2	3	4	5	0	1	0	1	2	3	2	3	4	5

with period 9. There is only one terminal 2, namely, 4, leading to $W_1 = \{4, 5, 6, 7, 14, 15, 16, 17, \ldots,\}$. On the other hand, no SG-4 has an SG-5 as a follower, so the sets W_2, $W_3, \ldots,$ become $\{8, 9\}$, $\{18, 19\}, \ldots$. Since $g(x) = 0$ or 1 implies that x has an even number of components in W_1, we may write

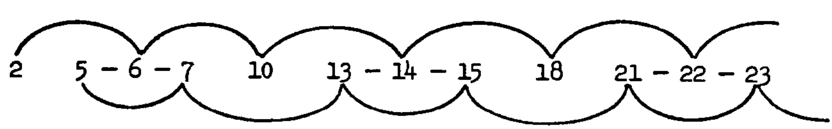
$Q = \{x\colon g(x) = 1$ and x has both or neither of the components in the sets $\{8, 9\}$, $\{18, 19\}, \ldots\}$,

$Q_0 = \{x\colon g(x) = 0$ and x has exactly one component in at least one of the sets $\{8, 9\}$, $\{18, 19\}, \ldots\}$.

4.10. $\{1, 2, 4, 6\}$ *and Others*

Of the remaining games with subtraction sets in Table I, namely, $\{1, 2, 4, 6\}$, $\{1, 2, 4, 5, 6\}$, $\{1, 2, 6, 7\}$, $\{2, 3, 4, 7\}$, $\{2, 3, 5, 7\}$, $\{2, 4, 5, 7\}$, $\{2, 4, 6, 7\}$, and $\{1, 2, 5, 6, 7\}$, all have SG-4s so that Theorem 4 does not apply, and all have two terminal 2s whose generated sets intersect so that Theorem 3 does not apply. Here we look briefly at $\{1, 2, 4, 6\}$ and see that the solution must be of an entirely different nature than those given in Theorems 3 and 4.

The reduced graph for $\{1, 2, 4, 6\}$ is



Since a position consisting of a single vertex of this graph is N, we first look at pairs. It is not difficult to see that $x = \{v_1, v_2\}$ is a P-position if and only if $\{v_1, v_2\}$ is one of the following four types for integers $k_1 \geqslant 0$ and $k_2 \geqslant 0$:

(1) $\{2+8k_1, 2+8k_2\}$, $k_1 \neq k_2$,
(2) $\{2+8k_1, 5+8k_2\}$,
(3) $\{6+8k_1, 6+8k_2\}$, $k_1 \neq k_2$,
(4) $\{5+8k_1, 7+8k_2\}$.

The first three types have SG-value 0, but the last has SG-value 6, showing that some entirely different approach is needed to solve this game.

For positions of the form $x = \{1, v_1, v_2\}$, the P-positions may be described as

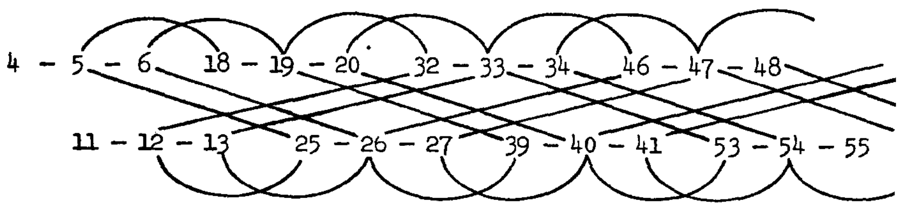
(1) $\{1, 2, 7\}$,
(2) $\{1, 5+8k_1, 5+8k_2\}$, $k_1 \neq k_2$,
(3) $\{1, 6+8k_1, 7+8k_2\}$, $k_1 \neq k_2$, $k_1 \neq k_2 + 1$,
(4) $\{1, 10+8k_1, 7+8k_1\}$ $(k_1 = k_2)$
(5) $\{1, 2+8k_1, 6+8k_2\}$, $k_1 \neq k_2$.

This completely describes the P-positions when there are at most two counters on the vertices of SG-values 2 or greater. One can see that there also exist triplet P-positions with components in V_1 such as $\{10, 14, 15\}$ and $\{7, 14, 18\}$, unlike any of the solutions found above for graphs with all SG-values 5 or less.

4.11. *Other Games*

Theorem 4 has been fairly successful at solving the subtraction games of Table I whose SG-sequence contains no values greater than 3. It is quite possible that a complete theory for such games exists. In this section, we give without proof the solution to two such games not solved by Theorem 4, that may give an indication of the direction that the complete theory will have to move. It would seem from these examples, however, that the complete theory is quite complex.

EXAMPLE 1. $\{1, 3, 4, 13, 20\}$. The reduced graph is



The terminal 2s generate $W_1 = \{4, 5, 6, 18, 19, 20, 39\}$ and $W_2 = \{11, 12, 13\}$, but the graph does not satisfy Condition C1 because of the annihilable pairs $W_3 = \{25, 26\}$ and $W_4 = \{32, 33\}$. However, THEOREM 5. *For the game* $\{1, 3, 4, 13, 20\}$, *let*

$$Q_1 = \{x: g(x) = 1 \text{ and } x \in E(W_1, W_2, W_3, W_4)\},$$
$$Q_0 = \{x: g(x) = 0, x \text{ is not terminal and } x \nrightarrow Q_1\}$$

then $P = Q_1 \cup Q_0$.

The following lemma will help give a better description of the set Q_0.

LEMMA 3. *Let x be non-terminal with SG-value* 0. *Then* $x \rightarrow Q_1$ *if, and only if,*

(1) $x \in E(W_0, W_1, \ldots, W_4)$, *where* W_0 *consists of some annihilable pair in* R, *or*

(2) *for some* $1 \leqslant j \leqslant 4$ *and* $v \notin W_j$ *with a follower in* W_j, $x \in E(W_1, \ldots, W_j + \{v\}, \ldots, W_4)$.

EXAMPLE 2. $\{2, 4, 5, 6\}$. The reduced graph is given in Example 3 of Section 4.4. The terminal 2s lead to the sets $W_1 = \{4, 6\}$ and $W_2 = \{5, 7, 13, 15, 21, \ldots\}$, so that $R = \{12, 14, 20, 22, \ldots\}$. In the solution given below, it is seen that a P-position with many elements of R must be in Q_1 rather than Q_0 as in Theorem 4, thus indicating that the general solution must be quite complex.

THEOREM 6. *Let*

$$Q_0 = \{x: g(x) = 0 \text{ and } x \text{ has an odd number of components in both } W_1 \text{ and } W_2 \text{ and no components in } R\},$$
$$Q_1 = \{x: g(x) = 1 \text{ and } x \nrightarrow Q_0\}.$$

Then $P = Q_1 \cup Q_0$.

A better idea of the set Q_1 is given by the following lemma.

LEMMA 4. *Let* $g(x) = 1$. *Then* $x \rightarrow Q_0$ *if and only if*

(1) *x has an odd number of components in* $W_1 + \{12\}$ *and* W_2 *and no components in* $\{14, 20, 22, \ldots\}$, *or*

(2) *x has an odd number of components in* W_1 *and an even number of components in* W_2 *plus one component in R that can reach* W_2, *or*

(3) *x has an odd number of components in* W_1 *and* W_2 *plus an annihilable pair of R.*

In all examples solved when all SG-values are 3 or less, it has turned out to be the case that $P = Q_1 \cup Q_0$, where Q_1 is a set of positions of SG-value 1 and $Q_0 = \{x: g(x) = 0 \text{ and } x \not\to Q_1\}$. It is easy to conjecture that this should always be the case but difficult to see why. An equivalent conjecture is that all P-positions have SG-values zero or one.

References

1. A. S. Fraenkel and Y. Perl, Constructions in combinatorial games with cycles, *in* "Infinite and Finite Sets," Colloquia Mathematica Societatis János Bolyai No. 10, pp. 667–699, Keszthely, Hungary, 1973.
2. A. S. Fraenkel and U. Tassa, Strategy for a class of games with dynamic ties, *Comput. Math. w. Appl.* **1** (1975), 237–254.
3. J. H. Conway, Loopy games, *in* "Advances in Graph Theory" (Ann. of Discrete Math. No. 3) (B. Bollobás, Ed.), North–Holland, Amsterdam, 1978.
4. J. H. Conway, "On Numbers and Games," Academic Press, New York/London, 1976.
5. J. A. Flanigan, Slow joins of loopy games, *J. Combin. Theory Ser. A* **34** (1983), 46–59.
6. C. L. Bouton, Nim, a game with a complete mathematical theory, *Ann. Math.* **3** (1901), 35–39.
7. R. Sprague, Über mathematische Kampfspiele, *Tohoku Math. J.* **41** (1936), 438–444.
8. P. M. Grundy, Mathematics and games, *Eureka* **2** (1939), 6–8.
9. T. S. Ferguson, On sums of graph games with last player losing, *Int. J. Game Theory* **3** (1975), 159–167.
10. A. S. Fraenkel, Combinatorial games with an annihilation rule, *in* "Proceedings Symposia in Applied Mathematics Vol. 20," pp. 87–91, Amer. Math. Soc., Providence, R. I., 1974.
11. A. S. Fraenkel and Y. Yesha, Theory of annihilation games, *Bull. Amer. Math. Soc.* **82** (1976), 775–777.
12. A. S. Fraenkel, U. Tassa, and Y. Yesha, Three annihilation games, *Math. Mag.* **51** (1978), 13–17.
13. R. B. Austin, "Impartial and Partisan Games," M. Sc. thesis, University of Calgary, Alberta, Canada, 1976.
14. E. R. Berlekamp, J. H. Conway, and R. K. Guy, "Winning Ways," Academic Press, New York/London, 1982.